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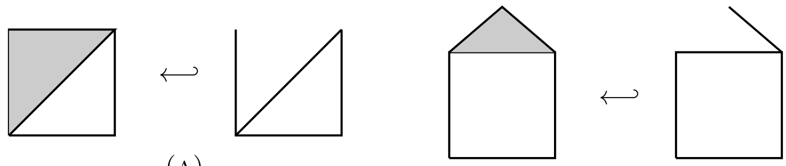
Claudia Scheimbauer
Eilind Karlsson

Algebraic Topology – Exercise 10

Note: Exercise session tomorrow (Friday 10.01) is in Room 02.06.011

(1) Compute the homology of the below four Δ-complexes.

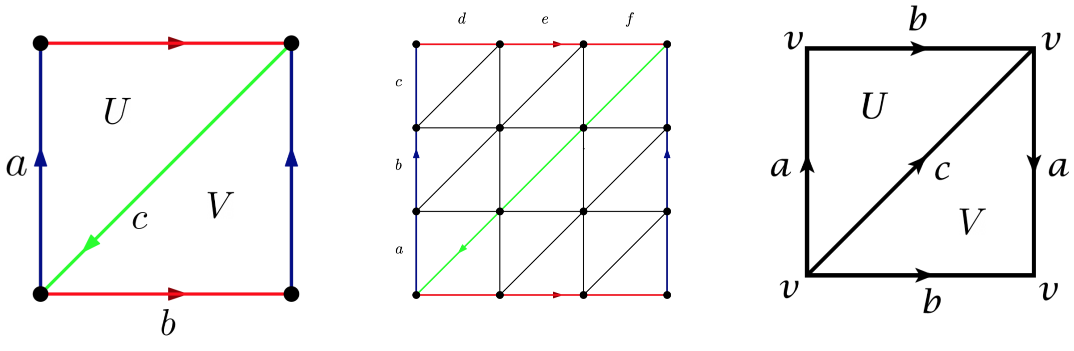


(A) (B)

(2) (a) Compute the simplicial homology groups of the torus using the Δ-complex structure given in Figure 1 (A) below.

(b) Consider a different Δ-complex structure on the torus by subdividing each triangle into several triangles as shown in Figure 1 (B)[1]. Compute the simplicial homology groups of the torus again using this Δ-complex structure and compare it to the result in a). Do you think the simplicial homology depends on the Δ-structure of a space X?

(c) Compute the simplicial homology groups of the Klein bottle using the Δ-complex structure given in Figure 1 (C) below.



(A) (B) (C)

Figure 2

[1]Note that this is even a triangulation of the torus.

(3) (a) Recall from class the semi-simplicial complex structure on Δ^n. The associated semi-simplicial set $\Delta^n_\bullet$ is given by $\Delta^n_k = \mathrm{Hom}_{\Delta_{\mathrm{inj}}}\big([\mathrm{k}],[\mathrm{n}]\big)$. What is the geometric realization of $\Delta^n_\bullet$?

(b) Consider $X_0 = \{x\}$, $X_1 = \{\tau\}$, $X_n = 0$ for $n \geqslant 2$, with $d_1(\tau) = x = d_0(\tau)$. What is the geometric realization of this semi-simplicial set?

(c) Let $X_0 = \{x_1, x_2, x_3\}$, $X_1 = \{\tau_1, \tau_2, \tau_3\}$, $X_2 = \{\sigma_1, \sigma_2\}$ and $X_n = 0$ for $n \geqslant 3$. The two maps $d_{0,1} : X_1 \to X_0$ are given by $d_0(\tau_1) = x_2 = d_1(\tau_2)$, $d_0(\tau_2) = x_3 = d_0(\tau_3)$, $d_1(\tau_3) = x_1 = d_1(\tau_1)$. The three maps $d_{0,1,2} : X_2 \to X_1$ are given by $d_0(\sigma_1) = \tau_2 = d_0(\sigma_2)$, $d_1(\sigma_1) = \tau_3 = d_1(\sigma_2)$ and $d_2(\sigma_1) = \tau_1 = d_2(\sigma_2)$. What is the geometric realization of this semi-simplicial complex?

(4) (a) Show that any morphism in Δ_{inj} can be written as a composition of face inclusions $d^i : [m] \to [m+1]$ which is defined by

$$d^i(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geqslant i \end{cases}$$

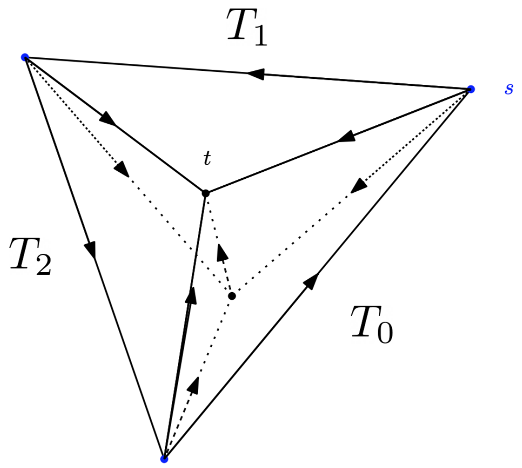
(b) Is there an algorithm for finding a unique composition of face inclusions corresponding to an arbitrary morphism g in Δ_{inj}?

(5) Construct a 3-dimensional Δ-complex X^2 from 3 tetrahedra T_0, T_1, T_2 by the following two steps. First arrange the tetrahedra in a cyclic pattern as in the below figure, so that each T_i shares a common vertical face with its two neighbors T_{i-1} and T_{i+1}, subscripts taken mod(n). Then identify the bottom face(on the backside) of T_i with the top face(which is towards you) of T_{i+1} for each $i \in \{1, 2, 3\}$.

(a) Verify that the Δ-complex of X has exactly two vertices corresponding to s and t indicated in the figure.

(b) Show that the simplicial homology groups of X in dimensions $0, 1, 2, 3$ are $\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, 0, \mathbb{Z}$ respectively.

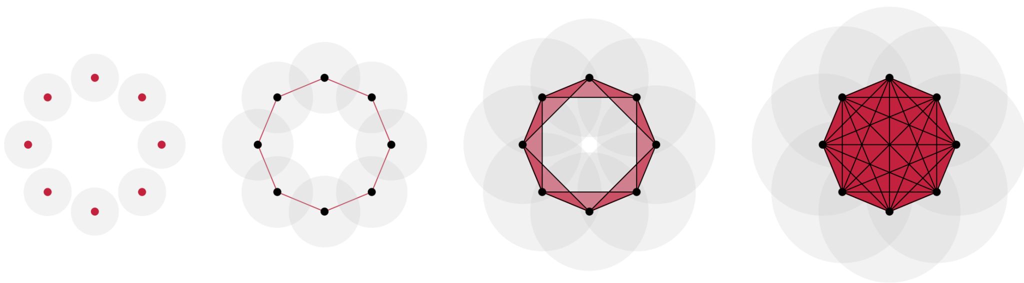
[2] The space X is an example of a *lens space*. For the interested reader the general definition of these spaces are given in Example 2.43 of Hatcher.



(6) One application of homology is to understand the "geometric" information of a data cloud[3]. Consider a finite set of points X. Instead of working with the set of points, one can induce a family of semi-simplicial complexes K_X^ϵ depending on the parameter $\epsilon \in \mathbb{R}$ in the following way: Use a distance measure $\mathrm{dist}(\cdot,\cdot)$ such as the Euclidean distance and connect the points $u, v \in X$ if $\mathrm{dist(u,v)} \leqslant \epsilon$, for varying ϵ. In addition, a k-simplex is added whenever all $k-1$-dimensional faces of the k-simplex are present.

(a) What is the fundamental group and the rank of its abelianization of the shaded area for each value of ϵ in the figure below?

(b) Compute the first homology of the below Δ-complex induced from X for the four values of ϵ.



[3]This is e.g. explained in more words in https://www.ams.org/journals/bull/2009-46-02/S0273-0979-09-01249-X/S0273-0979-09-01249-X.pdf.