winter semester 2019/20

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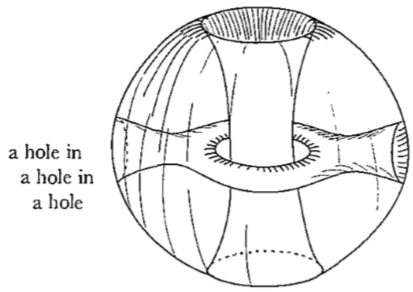
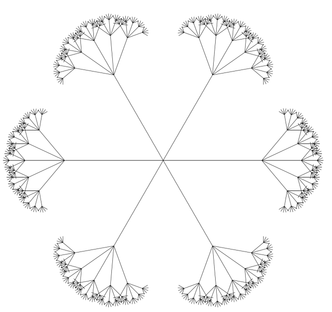
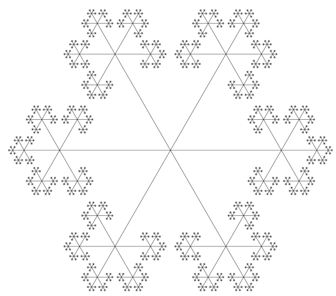
Algebraic Topology – Exercise 8

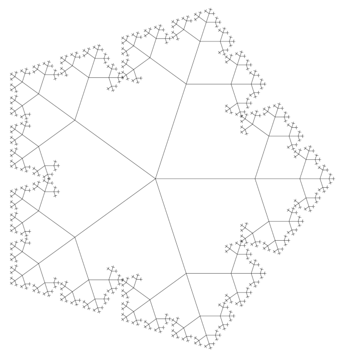
(1) What is the genus of the following surface?

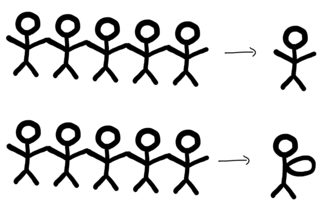


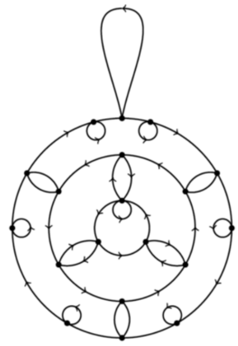
(2) Which of the following are non-trivial covering spaces? For which base spaces? Which ones can we restrict to obtain a covering?


(A)


(B)

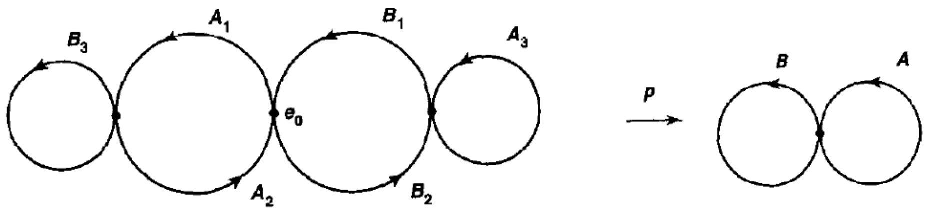

(C)


(D) (1) and (2)


(E)

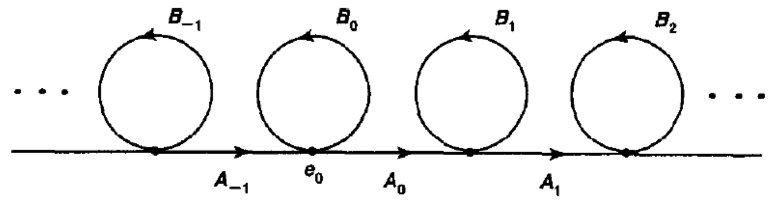
(3) Let $X = S^1 \vee S^1$ be the wedge of two circles.

(a) Determine the group of covering transformations of the covering of X indicated in the below figure.



If the basepoint in X is lifted to e_0, what does the loop A in X lift to in the covering space? What happens if you change to another lift of the basepoint?

(b) Repeat the above exercise for the below covering space and the universal covering of X (c.f. Exercise 4 on Sheet 6).



(4) (a) Find the fundamental group of $\mathbb{R}P^2$ using its universal cover. *Hint: see e.g. Exercise 6 on Sheet 6.*

(b) Compute the fundamental group of $\mathbb{R}P^2$ using Seifert-van Kampen on the proper labelling scheme which gives the projective plane.

(5) (a) Show that $abcda^{-1}b^{-1}c^{-1}d^{-1} \sim aba^{-1}b^{-1}cdc^{-1}d^{-1}$ by using elementary operations on labelling schemes. Draw the corresponding surface.

(b) Let X be a space obtained by pasting the edges of a polygonal region labelled by a proper labelling scheme. Show that X is homeomorphic to exactly one of the spaces in the following list: $S^2, P^2, K, \Sigma_g, \Sigma_g \# P^2, \Sigma_g \# K$, where K is the Klein bottle and $g \geqslant 1$.

Definition. A *triangulation of a surface* X is a collection of triangles $A_1, .., A_n$ in X s.t. $X = A_1 \cup ... \cup A_n$ and for $i \neq j$ the intersection $A_i \cap A_j$ is either empty, a vertex of both triangles or an edge of both triangles.

(6) Given a polyhedron the Euler characteristic $\chi(X)$ is defined as $\chi(X) := V - E + F$, where V, E, and F are the number of vertices, edges and faces, respectively. Choose a triangulation of Σ_g for $g = 0, 1, 2$ and compute the Euler characteristic $\chi(\Sigma_g)$. Compare with others: if they chose a different triangulation, did they get a different number?
Extra: How can you find a triangulation of the surface starting from a pasting scheme? Use this to compute the Euler characteristic of P_k as well. *Hint: Subdivide.*

(7) *Reading exercise.* Read "A Note on the Universal Covering Space of a Surface" by G. W. Knutson, with emphasis on understanding the first part. The (short) paper can be found at: `https://www.jstor.org/stable/pdf/2317755.pdf`.