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Algebraic Topology – Exercise 9

(1) Let Δ^n be the standard topological n-simplex, i.e.

$$\Delta^n = \{(t_0, ..., t_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} t_i = 1, t_i \geqslant 0\} \tag{1}$$

endowed with the subspace topology of $\mathbb{R}^{n+1}$. For $0 \leqslant i \leqslant n$ we define the coface maps

$$\delta_i : \Delta^{n-1} \hookrightarrow \Delta^n, \quad (t_0, .., t_{n-1}) \mapsto (t_0, .., t_{i-1}, 0, t_i, ..., t_{n-1})$$

and for $0 \leqslant i < n$ we define the codegeneracy maps

$$\sigma_i : \Delta^n \to \Delta^{n-1}, \quad (t_0, ..., t_n) \mapsto (t_0, ..., t_{i-1}, t_i + t_{i+1}, t_{i+2}, ..., t_n).$$

Check that the coface and codegeneracy maps together satisfy the cosimplicial identities:

$$\begin{cases} \delta_j \circ \delta_i = \delta_i \circ \delta_{j-1}, & 0 \leqslant i < j \leqslant n \\ \sigma_j \circ \delta_i = \delta_i \circ \sigma_{j-1}, & 0 \leqslant i < j \leqslant n \\ \sigma_j \circ \delta_j = id = \sigma_j \circ \delta_{j+1}, & 0 \leqslant j \leqslant n \\ \sigma_j \circ \delta_i = \delta_{i-1} \circ \sigma_j, & 1 \leqslant j+1 < i \leqslant n \\ \sigma_j \circ \sigma_i = \sigma_i \circ \sigma_{j+1}, & 0 \leqslant i \leqslant j \leqslant n \end{cases}$$

(2) Let $n \in \mathbb{N}_0$ and $k \in \{0, ..., n\}$ be arbitrary. Let $E_n \subset \mathbb{R}^{n+1}$ be the unique n-dimensional affine subspace of $\mathbb{R}^{n+1}$ that contains the standard basis vectors $(e_i)_{i=0,...,n}$. We let $\partial\Delta^n$ denote the topological boundary of the standard topological n-simplex Δ^n, seen as a subspace of E_n. Further, the *k-th horn* of Δ^n is defined as the union

$$\Lambda^n_k := \bigcup_{i \in \{0,..,n\} \setminus \{k\}} \delta_i(\Delta^{n-1}) \quad \subset \Delta^n$$

of the images of the coface maps δ_i from Exercise (1) for $i \in \{0, .., n\} \setminus \{k\}$.

(a) Draw Δ^n and Λ^n_k for $n = 0, 1, 2$ and all possible $k \in \{0, .., n\}$. Explain why Δ^n and Λ^n_k are intuitive choices of notation.

(b) Give explicit expressions of the form of Equation (1) for $\partial\Delta^n$ and for Λ^n_k.

(c) Show that Δ^n deformation retracts onto any of its (co)faces $\delta_k\Delta^{n-1}$. Do so by constructing a deformation retraction $h : [0, 1] \times \Delta^n \to \Delta^n$ whose restriction to Λ^n_k yields a homeomorphism $h_1|_{\Lambda^n_k} : \Lambda^n_k \to \delta_k\Delta^{n-1}$.

(d) Let X be a topological space. A Λ^n_k-horn on X is a continuous map $\alpha : \Lambda^n_k \to X$. Use the statement of part (c) to prove that any Λ^n_k-horn $\alpha : \Lambda^n_K \to X$ on X can be extended to an n-simplex $\hat{\alpha} : \Delta^n \to X$ on X. [1]

Definition. A *chain complex* of abelian groups is a sequence of abelian groups, $(A_n)_{n\in\mathbb{Z}}$, together with a differential, i.e. homomorphisms

$$\partial_n : A_n \longrightarrow A_{n-1}, \qquad \forall n \in \mathbb{Z},$$

such that $\partial_{n-1} \circ \partial_n = 0$.

Definition. Let (C_*, ∂) be a chain complex. Define

$$Z_n(C) := \ker(\partial_n : A_n \to A_{n-1}) \quad \text{and} \quad B_n(C) := \operatorname{im}(\partial_{n+1} : A_{n+1} \to A_n).$$

We say that $Z_n(C)$ is the group of *n-cycles*, and $B_n(C)$ is the group of *n-boundaries.* Define

$$H_n(C) := Z_n(C)/B_n(C).$$

We say that $H_n(C)$ is the *nth homology group* of the chain complex (C_*, δ).

(3) (a) Let (C_*, ∂) be an arbitrary chain complex. Why is $B_n(C)$ a subset of $Z_n(C)$?

(b) Consider the chain complex

$$C_n = \begin{cases} \mathbb{Z} & n = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

Here, the only non-zero differential is ∂_1, given by multiplication with $d \in \mathbb{N}$. Compute the homology groups for the chain complex C_*.

(c) What are the homology groups of the below chain complex?

$$C_* := \ldots \xleftarrow{2\cdot(-)} \mathbb{Z}/4\mathbb{Z} \xleftarrow{2\cdot(-)} \mathbb{Z}/4\mathbb{Z} \xleftarrow{2\cdot(-)} \ldots$$

(4) (a) Given a Δ-structure on X, recall that X_n is the set of n-simplices of the Δ-structure and $d_i \colon X_n \to X_{n-1}$, $\sigma \mapsto d_i(\sigma)$. Show that $d_i d_j = d_{j-1} d_i$ for $i < j$.

(b) Define

$$C_n^{\Delta}(X) := \mathbb{Z}X_n$$

and $\partial_n \colon C_n^{\Delta}(X) \to C_{n-1}^{\Delta}(X)$ given by $\partial_n = \sum_{i=0}^{n} (-1)^i d_i$. Here $d_i : \mathbb{Z}X_n \to \mathbb{Z}X_{n-1}$ is the unique map extending the face map $d_i : X_n \to X_{n-1}$ by linearity. Show that (C_*^{Δ}, ∂) is a chain complex, i.e. show that $\partial^2 = 0$.

(c) Compute the homology of $C_n^{\Delta}(\mathbb{R}P^2)$ using the Δ-structure on $\mathbb{R}P^2$ given by adding a diagonal in the labelling scheme *abab*.

(5) *Christmas reading exercise.* Choose one (or more) of the articles on the Algebraic Topology webpage under "Further reading" that sparks your interest to explore over the break.

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[1] The insight from part (c) can be used to concatenate 1-simplices "up to 2-simplices". That is, given two 1-simplices $\alpha_0, \alpha_2 : \Delta^1 \to X$ such that $\partial_1 \alpha_0 = \partial_0 \alpha_2$ show that there exists some 2-simplex $\beta : \Delta^2 \to X$ such that $\partial_j \beta = \alpha_j$ for $j = 0, 2$. We may then call $\alpha_1 := \partial_1 \beta : \Delta^1 \to X$ a (not the!) concatenation of α_0 and α_2; in general the 1-simplex $\alpha_1 : \Delta^1 \to X$ depends on the choice of β.