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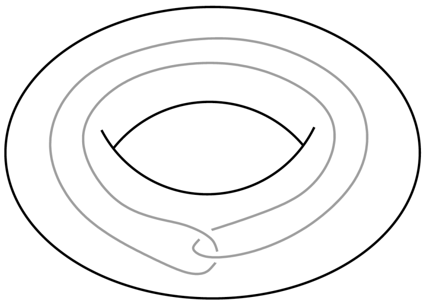
Algebraic Topology – Exercise 5

(1) Show that there is no retraction[1] $r : X \to A$ in the following cases:

(a) $X = D^2$ and $A = S^1$,

(b) $X = S^1 \times D^2$ and $A = S^1 \times S^1$,

(c) $X = S^1 \times D^2$ and A the following closed path in X:



(2) (a) Show that for every $x_0 \in X, y_0 \in Y$ there is an isomorphism of fundamental groups

$$\pi_1(X \times Y, (x_0, y_0)) \longrightarrow \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

(b) Compute $\pi_1(SL_2(\mathbb{R}), M)$, for any $M \in SL_2(\mathbb{R})$.

(3) (a) Let A be a subset of X containing a point in each path-connected component of X. Show that there is an equivalence of categories $\Pi_{\leqslant 1}(X, A) \simeq \Pi_{\leqslant 1}(X)$.

(b) Show that the following diagram is a pushout of groupoids[2]

$$\begin{tikzcd} \{0,1\} \arrow[r] \arrow[d] & \{0\} \arrow[d] \\ \pi_1([0,1], \{0,1\}) \arrow[r] & B\mathbb{Z} \end{tikzcd}$$

Which geometric argument does this remind you of?

[1] Let $\iota : A \hookrightarrow X$ be an embedding. A continuous map $r : X \to A$ is a *retraction if* $r \circ \iota = id_A$

[2] Here $\{0, 1\}$ is a groupoid with two objects $0, 1$, and two morphisms id_0, id_1.

(4) (a) Let $n > 2$ and let X be a path-connected topological space. Show that if Y is obtained from X by attaching n-cells, then for any $x_0 \in X$ the induced map $X \to Y$ induces an isomorphism $\pi_1(X, x_0) \cong \pi_1(Y, x_0)$.

(b) Show that if X is a path-connected cell complex, then the inclusion of the 2-skeleton $X^2 \hookrightarrow X$ induces, for any $x_0 \in X^2$, an isomorphism $\pi_1(X^2, x_0) \cong \pi_1(X, x_0)$.

(c) Recall from Exercise 5 on Sheet 2 that real projective n-space $\mathbb{R}P^n$ can be constructed from $\mathbb{R}P^{n-1}$ by attaching an n-cell. Conclude that this inclusion induces isomorphisms

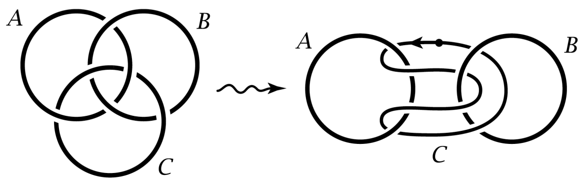
$$\pi_1(\mathbb{R}P^2) \cong \pi_1(\mathbb{R}P^3) \cong ... \cong \pi_1(\mathbb{R}P^\infty).$$

Extra: If Y is obtained from X by attaching 2-cells, what can we then say about the map $\pi_1(X, x_0) \to \pi_1(Y, x_0)$ induced by the map $X \to Y$?

(5) (a) Find the fundamental group of $\mathbb{R}^3 \backslash A$, where A is a single circle in $\mathbb{R}^3$.

(b) Find the fundamental group of $\mathbb{R}^3 \backslash (A \cup B)$, where A and B are two disjoint circles. Consider both the case where A and B are unlinked or linked.

(c) By viewing one of the circles in the Borromean rings as a loop in the complement of the other two (illustrated below), conclude that the Borromean rings are not unlinked in $\mathbb{R}^3$.



Definition. A sequence of groups G_i and group homomorphisms f_i

$$G_0 \xrightarrow{f_1} G_1 \xrightarrow{f_2} G_2 \xrightarrow{f_3} ... \xrightarrow{f_n} G_n$$

is called *exact* if the image of each homomorphism is equal to the kernel of the next, i.e. $\mathrm{im}(f_k) = \ker(f_{k+1})$ for all $k \in \{1, n-1\}$.

(6) If we assume that all the fundamental groups appearing in the Seifert-van Kampen theorem are abelian, one has an exact sequence

$$\pi_1(U \cap V, x) \to \pi_1(U, x) \oplus \pi_1(V, x) \to \pi_1(U \cup V, x) \to 0.$$

Construct this exact sequence, and give an example illustrating that the left homomorphism is not injective.