winter semester 2019/20 Claudia Scheimbauer

Algebraic Topology – Exercise 1

Exercise class: Fri 8:30–10:00 Room 00.07.014 will be given by Eilind Karlsson, first exercise: Friday, Oct. 25

Website with further material, including exercise sheets:
`https://www.groups.ma.tum.de/algebra/scheimbauer/` ⇒ Lehre ⇒ Algebraic Topology

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(1) Decide whether or not the following sets $\mathcal{O}$ define a topology on X, where

 (a) X is arbitrary, $S \subset X$ is a subset and $\mathcal{O} := \{U \subset X \mid S \subset U\} \cup \{\varnothing\}$.

 (b) X is arbitrary, $S \subset X$ is a subset and $\mathcal{O} := \{U \subset X \mid U \cap S \neq \varnothing\} \cup \{\varnothing\}$.

 (c) $X = \mathbb{R}$ and $\mathcal{O} := \{(-\infty, b) \mid b \in S \cup \{\infty\}\} \cup \{\varnothing\}$, for a subset $S \subset \mathbb{R}$.

(2) Let $(X, \mathcal{T})$ be a topological space. Prove that the closed subsets in $(X, \mathcal{T})$ satisfy the following axioms:

(A1) $\varnothing \in \mathcal{A}, X \in \mathcal{A}$

(A2) If $A_1, \ldots, A_n \in \mathcal{A}, n \in \mathbb{N}$, then $A_1 \cup \cdots \cup A_n \in \mathcal{A}$.

(A3) If for every $i \in I, A_i \in \mathcal{A}$, then $\bigcap_{i \in I} A_i \in \mathcal{A}$.

and conversely, given $\mathcal{A} \subseteq \mathcal{P}(X)$ which satisfies (A1)-(A3),

$$\mathcal{T} = \{X \backslash A : A \in \mathcal{A}\}$$

defines a topology on X.

Definition. Let X be a set. A *basis for a topology on* X is a collection of subsets $\mathcal{B}$ of X satisfying

(B1) $\bigcup_{B \in \mathcal{B}} B = X$, and

(B2) if $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

(3) (a) Let $\mathcal{B}$ be a basis for a topology on a set X. We say that U is open if for every $x \in U$, there is a $B \in \mathcal{B}$ such that $x \in B \subseteq U$. Prove that the collection of open sets $\mathcal{T}_{\mathcal{B}}$ forms a topology. It is called the *topology generated by the basis* $\mathcal{B}$.

 (b) Let $p \in \mathbb{N}$ be prime. For every pair $(z, n) \in \mathbb{Z} \times \mathbb{N}$, let

$$U_n(z) = \{z + mp^n : m \in \mathbb{Z}\}.$$

 Show that the family of these subsets of $\mathbb{Z}$ is a basis of a topology on $\mathbb{Z}$.

Definition. Let X be a topological space and $A \subset X$. Then $\bar{A} = \bigcap_{\substack{C \in \mathcal{A}, \\ A \subset C}} C$ is called the *closure* of A, and $\partial A := \overline{A} \cap \overline{X \backslash A}$ is called the *boundary* of A.

(4) Prove that for any subset A of a topological space

(a) A is open $\Leftrightarrow A = \overline{A} \backslash \partial A$,

(b) A is open and closed $\Leftrightarrow \partial A = \varnothing$.

(5) (a) Let X, Y be topological spaces, and $f : X \to Y$. Show that if $\{x\} \subset X$ is open, then f is continuous at x.

(b) Let $-\infty \leqslant a < b \leqslant \infty$. Show that (a, b) is homeomorphic to $(0, 1)$, and if a, b are finite, then $[a, b]$ is homeomorphic to $[0, 1]$.

(c) Give an example of a function $\mathbb{R} \to \mathbb{R}$ which is continuous at exactly one point.

(6) Let Y be a set, $(X_i, \mathcal{T}_i), i \in I$, be topological spaces and $f_i : Y \to X_i$ be maps.

(a) Let $\mathcal{S} = \bigcup_{i \in I} f_i^{-1}(\mathcal{T}_i)$. Show that the collection of finite intersections of elements in $\mathcal{S}$ is a basis on Y which generates a topology such that $\mathcal{T}$ is the coarsest topology on X such that all $f_i : (Y, \mathcal{T}) \to (X_i, \mathcal{T}_i)$ are continuous, and is characterized by the following property:

If $(Z, \mathcal{V})$ is a topological space and $f : Z \to Y$, then $f : (Z, \mathcal{V}) \to (Y, \mathcal{T})$ is continuous iff for every $i \in I$, $f_i \circ f : (Z, \mathcal{V}) \longrightarrow (X_i, \mathcal{T}_i)$ is continuous.

This is called the *initial topology* with respect to the f_i.

(b) Let $A \subset X$ be a subset of a topological space X. Show that if we take $I = \{0\}$, $f_0 : A \hookrightarrow X$, the initial topology on A with respect to f_0 is the subspace topology on A.

(c) Let X_1, X_2 be topological spaces and, as a set, $Y = X_1 \times X_2$. Show that the initial topology on Y with respect to the projections

$$f_1 : X_1 \times X_2 \longrightarrow X_1, \quad f_2 : X_1 \times X_2 \longrightarrow X_2$$

is the product topology on $Y = X_1 \times X_2$. More generally, show that this holds for any family of topological spaces $(X_i)_{i \in I}$ and the projections $f_i : \prod_{j \in I} X_j \to X_i$.

(d) Let $X := \{0, 1\}$ with the discrete topology. Show that $\prod_{n=1}^{\infty} X$ with the product topology is not discrete.

(7) Show that the following are categories:

(a) $\mathbb{N}$, which has just one object $*$, $\mathrm{Hom}_{\mathbb{N}}(*, *) = \mathbb{N}$, and composition is addition;

(b) $\mathrm{V}\textsc{ect}_{\mathbb{K}}$, which has vector spaces as its objects, linear maps as its morphisms, and composition is the composition of linear maps;

(c) given a partially ordered set $(P, \leqslant)$, take P as the objects, and set

$$\mathrm{Hom}_P(a, b) = \begin{cases} * & \text{if } a \leqslant b, \\ \varnothing & \text{otherwise.} \end{cases}$$