Algebraic Topology – Exercise 2

Products, quotients, pushouts

(1) The *Cantor set* is the subset of $\mathbb{R}$ defined by $\mathcal{C} := \left\{ \sum_{n=1}^{\infty} \frac{a_n}{3^n} \mid \forall n : a_n \in \{0,2\} \right\}$, i.e. $\mathcal{C}$ is the set of all real numbers in $[0,1]$ that have a ternary representation without any digit 1. More pictorially, the Cantor set can be obtained in the follwing way: Let $\mathcal{C}_0 = [0,1]$ Deleting the open middle third, we obtain $\mathcal{C}_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Deleting the open middle thirds from each of the remaining intervals, we obtain $\mathcal{C}_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$, etc. Then

$$\mathcal{C} = \bigcap_{n=0}^{\infty} \mathcal{C}_n.$$

If we endow the Cantor set with the subspace topology, show that it is homeomorphic to $\{0,1\}^{\mathbb{N}}$, where $\{0,1\}$ has the discrete topology and the product is endowed with the product topology.

(2) Let Y be a set, $(X_i, \mathcal{T}_i), i \in I$, be topological spaces and $f_i : X_i \to Y$ be maps.

(a) Show that $\mathcal{T} = \{O \subseteq Y \mid \forall i \in I : f_i^{-1}(O) \in \mathcal{T}_i\}$ is a topology on Y such that $\mathcal{T}$ is the finest topology on X such that all $f_i : (X_i, \mathcal{T}_i) \to (Y, \mathcal{T})$ are continuous and is characterized by the following property:

If $(Z, \mathcal{V})$ is a topological space and $f : Y \to Z$, then $f : (Y, \mathcal{T}) \to (Z, \mathcal{V})$ is continuous iff for every $i \in I$, $f \circ f_i : (X_i, \mathcal{T}_i) \longrightarrow (Z, \mathcal{V})$ is continuous.

This is called the *final topology* with respect to the f_i.

(b) Let X be a set and $\mathcal{T}_i$ be topologies on X. What is the final topology on X with respect to the maps $\mathrm{id} : (X, \mathcal{T}_i) \to X$?

(c) Let $X := \{0,1\}$ with the discrete topology. Show that $\coprod_{n=1}^{\infty} X$ with the coproduct topology is discrete.

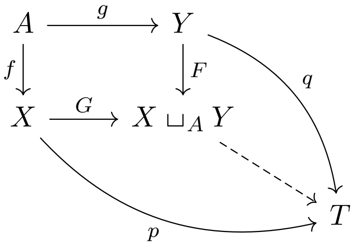
(3) Use the universal properties of the product and the quotient to construct a continuous map

$$(\mathbb{R} \times \mathbb{R}) / (\mathbb{Z} \times \mathbb{Z}) \longrightarrow \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}.$$

Show that this is a homeomorphism.

(4) (a) Show the universal property of the pushout of $f \colon A \to X$ and $g \colon A \to Y$: Let $p \colon X \to T$ and $q \colon Y \to T$ continuous maps such that $p \circ f = q \circ g$. Then there is

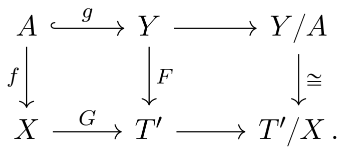
a unique map $X \sqcup_A Y \to T$ such that the following diagram commutes:



Conclude that any other space T' with the same universal property is homeomorphic to $X \sqcup_A Y$. We call any such T' a pushout of f and g.

(b) Show that $V \subset X \sqcup_A Y$ is open iff $G^{-1}(V)$ is open in X and $F^{-1}(V)$ is open in Y. Same statement for closed instead of open.

(c) Show that if g is a (closed) embedding[1] , then G also is. Moreover, show that in this situation, F induces a homeomorphism of quotient spaces $\bar{F}\colon Y/A \to T'/X$



(d) Show that if g is a homeomorphism, then G is an homeomorphism.

(5) (a) Show that the following descritions of real projective n-space are equivalent:

$$\left(\mathbb{R}^{n+1}\backslash 0\right)/(x \sim kx) \cong S^n/(x \sim -x) \cong D^n/(x \sim -x \text{ for } x \in \partial D^n).$$

(b) Use the last description above and induction to construct a cell structure on $\mathbb{RP}^n$. *Hint: Use that ∂D^n with antipodal points identified is $\mathbb{RP}^{n-1}$.*

Extra:

(6) (a) Let $X = \mathbb{R}^{\mathbb{N}}$ and let $\mathcal{O}_{\text{prod}}, \mathcal{O}_{\text{box}}$ and $\mathcal{O}_{\text{u}}$ be the product topology, the box topology and the uniform topology (see below) on X, respectively. For which of these topologies is the map $\mathbb{R} \to X, t \mapsto (t, t, \ldots)$ continuous?

(b) Let $\iota_n : \mathbb{R}^n \hookrightarrow \mathbb{R}^{\mathbb{N}}$ be the inclusion of the first n coordinates, and let $\mathbb{R}^n$ be endowed with the standard (Euclidean) topology. Show that the final topology on $\mathbb{R}^{\mathbb{N}}$ with respect to the ι_n is strictly finer than the box topology.

Definition: The *uniform topology* on $\mathbb{R}^{\mathbb{N}}$ is the topology $\mathcal{O}_{\text{u}}$ generated by the basis

$$\mathcal{B}_{\text{u}} := \left\{B_r((x_n)) = \{(y_n) \in \mathbb{R}^{\mathbb{N}} \mid \sup_{n \in \mathbb{N}} |y_n - x_n| < r\}, \text{for } r > 0 \text{ and } (x_n) \in \mathbb{R}^{\mathbb{N}}\right\}.$$

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[1]An *embedding* is a continuous map which is a homeomorphism onto its image (which is endowed with the subspace topology). It is closed if its image is closed in Y.

More operations on and properties of topological spaces

(7) (a) Construct a cell structure on S^n for every n such that the inclusions as equators $S^0 \subset S^1 \subset S^2 \subset \cdots \subset S^n$ are all inclusions of cell complexes.

(b) Show that the suspension of the n-sphere is an $(n+1)$-sphere, $SS^n = S^{n+1}$. Think about the cell structures of each side.

(c) Let X be a cell complex with n-skeleton X^n and attaching maps $\varphi_\alpha : S^{n-1} \to X^{n-1}$. Convince yourself that

$$X^n/X^{n-1} \cong \bigvee_{\alpha \in I} S^n.$$

(d) What is $S^1 \wedge S^1$? More generally, what is $S^n \wedge S^k$?

(8) Show that the following are topological properties, i.e. if $X \cong Y$ and X has the property, then Y also has the property:

cell complex, connected, path-connected, locally path-connected, Hausdorff, compact

Moreover, show that the set of path components is a topological invariant

(9) (a) Show that every connected component of a space X is closed in X.

(b) For X a space, let $x \sim y$ if x and y are in the same connected component. Show that

$$X = \bigsqcup_{C \in X/\sim} C \Leftrightarrow X/\sim \text{ is discrete}$$

(c) Conclude that if $X/\sim$ is finite, then $X = \bigsqcup_{C \in X/\sim} C$.

(10) Show that if Y is discrete and X is locally path-connected, then every map $g : \pi_0(X) \to Y$ induces a continuous map $f : X \to Y$. Conclude that in this case, there is a bijection

$$\mathrm{Hom}_{\mathrm{TOP}}(X, Y) \cong \mathrm{Hom}_{\mathrm{SET}}(\pi_0(X), Y).$$

Remark. In the literature, $\pi_0(X)$ sometimes is used for the connected components of X instead of the path components of X as we had defined it. If we assume that our space is locally path-connected, then they agree. In practice, and for most of the remainder of this class, spaces that are studied in algebraic topology are usually locally path-connected, so this issue does not arise very often.

(11) Prove the Lemma from class:

Lemma. *A topological space X is Hausdorff if and only if $\Delta = \{(x, x) : x \in X\} \subset X$ is closed.*