winter semester 2019/20 Claudia Scheimbauer

Algebraic Topology – Exercise 3

Compact, Hausdorff, proper, fiber bundles

(1) Prove the proposition from class:

Proposition. *Let X be compact and Y Hausdorff. Then any continuous map $f : X \to Y$ is proper.*

(2) Is $\mathbb{RP}^n$ compact or Hausdorff? Show that $\mathbb{RP}^1$ is homeomorphic to S^1.

(3) Let X be a topological space. The *Alexandroff compactification* or *one-point compactification* $\widehat{X}$ of X is defined as the disjoint union $\widehat{X} := X \sqcup \{p\}$, with the following topology:

$$U \subset \widehat{X} \text{ open} \iff \begin{cases} p \notin U \text{ and } U \subset X \text{ open, or} \\ p \in U \text{ and } X \backslash U \text{ closed and compact.} \end{cases}$$

(a) Is $\widehat{(-)}$ a functor from Top to itself?

(b) Prove that $\widehat{X}$ is compact and that the inclusion map $X \hookrightarrow \widehat{X}$ is a continuous embedding.

(c) Describe $\widehat{(0,1)}$, $\widehat{\mathbb{N}}$ and $\widehat{\mathbb{R}^n}$ geometrically.

(d) Show that if X is non-compact, then X is dense[1] in $\widehat{X}$, and if X is also connected, then $\widehat{X}$ is connected. Give an example of a disconnected space X, such that $\widehat{X}$ is connected.

(e) Show that $\widehat{X}$ is Hausdorff if, and only if, X is Hausdorff and locally compact.

(4) Show that the Hopf fiber bundle $p : S^3 \to S^2$ from class indeed is a fiber bundle. Recall that this was given by

$$\begin{aligned} p\colon S^3 \subset \mathbb{C}^2 &\longrightarrow S^2 \cong \mathbb{C} \cup \{\infty\} \\ (z_0, z_1) &\longmapsto z_0/z_1. \end{aligned}$$

Conclude that p is proper. Note: When writing $S^2 \cong \mathbb{C} \cup \{\infty\}$ we use that $S^2 \cong \widehat{\mathbb{R}^2}$ is the one-point compactification from exercise 3.

[1] $X \subset Y$ is *dense* if $\bar{X} = Y$.

(5) We define a topology on $\mathrm{Map}(X, Y)$ as follows: for $K \subset X$ compact and $U \subset Y$ open, let
$$M(K, V) = \{f \in \mathrm{Map}(X, Y)\colon f(K) \subset V\}.$$
Take the topology generated by the subbasis of all $M(K, V)$, i.e. a basis for the topology is given by finite intersections
$$\bigcap_{i=1}^{k} M(K_i, V_i).$$

(a) Show that for any $x \in X$, the evaluation
$$\mathrm{ev}_x\colon \mathrm{Map}(X, Y) \to Y, \qquad f \longmapsto f(x)$$
is continuous.

(b) Show that for $PX = \mathrm{Map}([0, 1], X)$, we have a homeomorphism
$$PX \times_X PX \xrightarrow{\cong} PX$$
given by composing paths. Here the fiber product on the left uses the maps $\mathrm{ev}_0, \mathrm{ev}_1\colon PX \to X$.

(6) A topological space X is called *contractible*, if the identity map $\mathrm{id}_X : X \to X$ is homotopic to a constant map.

(a) Show that $[0, 1]$ and $\mathbb{R}$ are contractible.

(b) Show that a contractible space is path connected.

Extra understanding:

(7) Prove a **characterization of locally compactness** by the following steps. First, show uniqueness of the one-point compactification: Let X be locally compact Hausdorff, with one-point compactification $Y = \widehat{X}$, and suppose that Y' is a compact Hausdorff space such that $X \subset Y'$ is a subspace and $Y' \backslash X$ is a single point. Show that the unique bijection that is the identity on X is a homeomorphism. Then conclude that a space X is homeomorphic to an open subspace of a compact Hausdorff space if and only if X is locally compact and Hausdorff.