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Algebraic Topology – Exercise 4

(1) Let A be a deformation retract of the topological space X, and let $x_0 \in A$.

(a) Show that the inclusion map

$$\iota : (A, x_0) \to (X, x_0)$$

induces an isomorphism of fundamental groups.

(b) Construct an explicit deformation retraction of $T \backslash \{p\}$, for $p \in T$, onto a graph consisting of two circles intersecting in a point.

(c) Show that the cone CX of X is contractible by constructing a deformation retract.

Definition. A functor $F : \mathcal{C} \to \mathcal{D}$ is an *equivalence of categories* if there is a functor $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms

$$F \circ G \Rightarrow \mathrm{id}_{\mathcal{D}} \qquad \text{and} \qquad G \circ F \Rightarrow \mathrm{id}_{\mathcal{C}}\,.$$

(2) Let X, Y be two topological spaces.

(a) Show that if $f : X \to Y$ is a homotopy equivalence, then $\Pi_{\leqslant 1}(f) : \Pi_{\leqslant 1}(X) \to \Pi_{\leqslant 1}(Y)$ is an equivalence of categories.

(b) Recall that any group G can be seen as a category with only one object $*$ and $\mathrm{Hom}(*, *) = G$. Show that if X is path-connected, then for any point $x \in X$ the inclusion

$$\pi_1(X, x) \to \Pi_{\leqslant 1}(X)$$

is an equivalence of categories. *Hint:* For the construction you need to choose a path from x to y.

Definition. A *topological group* G, is a topological space which is also a group, such that the group operations of product $G \times G \to G, (x, y) \mapsto xy$ and taking inverses $G \to G, x \mapsto x^{-1}$ are continuous.

(3) Show that if G is a topological group with identity element 1, then $\pi_1(G, 1)$ is abelian.

(4) Using the Seifert-van Kampen theorem, calculate the fundamental groups of the following topological spaces (at any base point):

(a) the n-sphere S^n for $n \geqslant 2$,

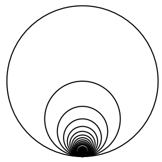
(b) the torus $T \cong S^1 \times S^1$,

(c) the bouquet of n circles, $B_n = \bigvee_{i=1}^{n} S^1$, for any n. What happens for $n = \infty$?

(5) The Hawaiian earring is defined as

$$H = \bigcup_{n=1}^{\infty} \{(x, y) \in \mathbb{R}^2 \mid x^2 + (y - \tfrac{1}{n})^2 = \tfrac{1}{n^2}\},$$

endowed with the subspace topology:



Show that H is not homeomorphic to $B_\infty = \bigvee_{n \in \mathbb{N}} S^1$ by

(a) using topological properties,

(b) comparing fundamental groups.