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Algebraic Topology – Exercise 6

(1) Let $p \times p : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1 =: T$ be the usual covering map of the torus and consider the path

$$\begin{aligned} f : [0,1] &\to T \\ t &\mapsto \big(\cos(2\pi nt) + i\sin(2\pi nt), \cos(2\pi kt) + i\sin(2\pi kt)\big), \end{aligned}$$

where $(n,k) \in \mathbb{Z} \times \mathbb{Z}$. Sketch f in the torus T for $n = 1, k = 2$, and find a lift $\widetilde{f}$ of f to $\mathbb{R} \times \mathbb{R}$. Same question for arbitrary $(n,k) \in \mathbb{Z} \times \mathbb{Z}$?

(2) (a) Let $\sigma : S^1 \to S^1$ be the map sending a point x in S^1 to its antipodal point $-x$. Find the homomorphism induced by σ on the fundamental group of S^1.

(b) View S^1 as the unit circle in $\mathbb{C}$. Let $\tau : S^1 \to S^1$ be the reflection across the real line. Find the homomorphism induced by τ on the fundamental group of S^1.

(3) (a) Let $F_n := \ast_n \mathbb{Z}$. Using covering maps, show that $F_2 = \mathbb{Z} * \mathbb{Z}$ has a subgroup isomorphic to F_3. More generally, for any $n \in \mathbb{N}$, show that F_2 has a subgroup isomorphic to F_n.

(b) Show that the free product of two non-trivial groups is neither abelian nor finite.

(c) Is the free product of two groups a free group? Give a proof or a counterexample.

(4) Let $B := S^1 \vee S^1$ be the topological space obtained by gluing together two copies of S^1 at a point. Find 3 non-homeomorphic covering spaces E_i, $i \in \{1,2,3\}$, where one of the E_i's is simply connected. *Hint:* Use the fact that paths in B can be lifted to paths in E.

(5) Let Σ_g denote the closed orientable surface of genus g. Find a 4-fold cover of Σ_1 that has an action of the cyclic group $\mathbb{Z}/4\mathbb{Z}$. Can you find a k-fold cover of Σ_g with an action of $\mathbb{Z}/k\mathbb{Z}$ for k and n arbitrary positive integers?

(6) Let G be a group. We say that an action of G on a topological space X is *properly discontinuous* if each $x \in X$ has a neighborhood U such that all the images gU for varying $g \in G$ are disjoint. In other words, $g_1 U \cap g_2 U \neq \varnothing$ implies $g_1 = g_2$.

(a) Let X be a connected topological space and G a group with a properly discontinuous on X. Show that the quotient map $X \to X/G$ is a covering map.

(b) Show that if X is simply-connected, locally path-connected and $x_0 \in X$, then $\pi_1(X/G, x_0) \cong G$.