winter semester 2019/20

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Algebraic Topology – Exercise 7

(1) (a) Compute the fundamental group of the torus $T = S^1 \times S^1$ using the Seifert van Kampen theorem for groups.

(b) Compute the fundamental group of the torus $T = S^1 \times S^1$ using the usual covering map $p \times p : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$, $(t, t') \mapsto (e^{2\pi i t}, e^{2\pi i t'})$.

(2) (a) Find the universal cover of $SO(2)$.

(b) Show that $SU(2)$ is a twofold cover of $SO(3)$. *Hint:* $SU(2)$ is isomorphic to a subgroup of the quaternions $\mathbb{H}$.

(c) *Plate trick.* Convince yourself that rotating an object with a ribbon connecting it to a fixed point by 2π does not return the system to its original state, but rotating by 4π does. This can for example be done by resting a small plate flat on the palm and performing two rotations of the hand while keeping the plate upright[1]. Conclude that $SU(2)$ in fact is a universal cover of $SO(3)$[2].

(3) A *graph* is a 1-dimensional CW complex. Let Γ be a connected graph with V vertices and E edges. The Euler characteristic is defined as $\chi(\Gamma) := V - E$. Show that $\pi_1(\Gamma)$ is a free group with rank $1 - \chi(\Gamma)$.

(4) Let $p : E \to \Gamma$ be a covering of a graph Γ. Show that E is a graph. Moreover, show that if Γ is locally finite then E is locally finite.

(5) Let G be a subgroup of a free group F. Using the two previous exercises and the following theorem from class, show that G is free.

Theorem. *Suppose B is path-connected, locally path-connected, and semilocally simply-connected. Then for every subgroup $H \subset \pi_1(B, b_0)$ there is a covering space $p : X_H \to B$ such that $p_*(\pi_1(X_H, \widetilde{x}_0)) = H$ for a suitably chosen basepoint $\widetilde{x}_0 \in X_H$.*

(6) Let B be a path-connected, locally path-connected and semilocally simply-connected topological space and let $b_0 \in B$. Let H be a subgroup of $\pi_1(B, b_0)$ and let $\mathcal{P}$ be the collection of all paths α in B with $\alpha(0) = b_0$. Define $E = \mathcal{P}/\sim$, where $\alpha \sim \beta$ iff $\alpha(1) = \beta(1)$ and $[\alpha * \beta^{-1}] \in H \subset \pi_1(B, b_0)$. Denote the equivalence class of α in E by $\alpha^{\#}$.

[1] Two great illustrations of this: `https://www.futilitycloset.com/2016/01/09/the-plate-trick/`

[2] This exercise is a special case of the general statement that for $n > 2$ the so-called Spin-group $Spin(n)$ is the universal cover of $SO(n)$. In particular $Spin(3) \simeq SU(2)$.

(a) Define

$$\mathcal{B}(U,\alpha) := \{(\alpha * \delta)^{\#} : \delta : [0,1] \to U, \delta(0) = \alpha(1)\},$$

where $\alpha \in \mathcal{P}$ and U is a path-connected neighborhood of $\alpha(1)$. Check that the sets $\mathcal{B}(U,\alpha)$ forms a basis for a topology on E

(b) Show that the map $p : E \to B, \alpha^{\#} \mapsto \alpha(1)$ is continuous and open.

(c) Conclude that p is a covering map by doing the following steps:

- For any $b_1 \in B$ choose a path-connected neighborhood $U \overset{\iota}{\hookrightarrow} B$ s.t. $\iota_* : \pi_1(U, b_1) \to \pi_1(B, b_1)$ is trivial (Why does such a neighborhood exist?). Show that $p^{-1}(U) = \bigcup_\alpha \mathcal{B}(U,\alpha)$, where the union is over all paths α from b_0 to b_1.
- Show that if $\beta^{\#} \in \mathcal{B}(U,\alpha)$ then $\mathcal{B}(U,\alpha) = \mathcal{B}(U,\beta)$, and conclude from this that the above union is in fact disjoint.
- Show that $p|_{\mathcal{B}(U,\alpha)}$ is a bijection.

(d) Let α be a path in B s.t. $\alpha(0) = b_0$, and denote by e_0 the equivalence class in E of the constant path at b_0 so we have $p(e_0) = b_0$. Show that

$$\widetilde{\alpha} : [0,1] \to E, \ c \mapsto \alpha_c^{\#}, \qquad \text{where} \quad \alpha_c(t) := \alpha(tc),$$

is a continuous map and defines a lift of α to E. Conclude that E is path-connected.

(e) Show that $H = p_*(\pi_1(E, e_0))$.

(f) Congratulations, you have now proved the Theorem above!