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Bordisms and TFTs - Exercise 13

(1) *Finite dimensional modules of a Hopf algebra*

Let H be a Hopf algebra with invertible antipode S. Show that any finite dimensional H-module has a left and right dual.

(2) *Tangle categories and link invariants*

(a) Let $F_{\mathcal{C}} : \mathrm{Tang}_1^{\mathrm{fr}}(V_1) \to \mathcal{C} = U_q\mathfrak{sl}_2$-mod be the functor defined at the end of Lecture 20[1]. Using the isomorphism

$$\begin{aligned} f : V_1 &\longrightarrow V_1^* \\ v_1 &\mapsto \hat{v}_{-1} \\ v_{-1} &\mapsto -q^{-1}\hat{v}_1, \end{aligned}$$

find $F_{\mathcal{C}}$ on the cup and cap, i.e. find the maps $F_{\mathcal{C}}(\subset) : \mathbb{C}(q) \to V_1 \otimes V_1$ and $F_{\mathcal{C}}(\supset) : V_1 \otimes V_1 \to \mathbb{C}(q)$. Compute the value of $F_{\mathcal{C}}(S^1) : \mathbb{C}(q) \to \mathbb{C}(q)$. Do you see any resemblance between this and the defining relations of the Kauffman bracket?

Definition. A *simple* module S is a non-zero module whose only submodules are $\{0\}$ and S.

(3) *Representations of Sweedler's Hopf algebra*

(a) Let H denote Sweedler's Hopf algebra defined in Exercise 2 on Sheet 10. Find all (up to isomorphism) simple H-modules.
Hint: If a module is 1-dimensional, it is simple.

(b) Prove that the tensor product of two simple H-modules is again simple.

(4) *Outlook/Reading exercise: Open-closed 2-TFTs*

There is a notion of *open-closed 2-TFTs* where one considers objects to be 1-dimensional (compact, oriented) manifolds possibly with boundary, i.e. finite disjoint unions of circles and intervals. In an analogous way to how 2-TFTs are given by commutative Frobenius algebras, one can show that open-closed 2-TFTs correspond to so-called *knowledgable Frobenius algebras.* Work through Example 4.1.14 in `https://www.math.uni-hamburg.de/home/schweigert/ws12/hskript.pdf` to understand this statement better.

[1]Initially the "hats" where wrong in the last expression, but it is now correct.