Summer semester 2021

Claudia Scheimbauer
Eilind Karlsson

# Bordisms and TFTs - Exercise 6

(1) *Euler TFTs*

Fix a non-zero number $\lambda \in \mathbb{C}$. Define an invertible TFT

$$F : \mathrm{Bord}_{n,n-1} \to \mathrm{Vect}_{\mathbb{C}}$$

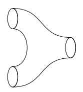
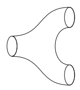
such that for any closed $n$-manifold $X$ we assign $F(X) = \lambda^{\chi(X)}$.

(2) *TFTs as categorified bordism invariants*

Show that an endomophism of $\mathbb{1}$ in $\mathrm{TFT}_n$ yields a bordism invariant. Which categories $\mathcal{C}$ should we use?

(3) *2-dimensional TFTs and Frobenius algebras*

Let $\mathcal{Z} : \mathrm{Bord}_{2,1} \to \mathcal{C}$ denote a 2-dimensional topological field theory with arbitrary target category. Set $A := \mathcal{Z}(S^1)$,

$m := \mathcal{Z}\Big($  $\Big)$ and $\Delta := \mathcal{Z}\Big($  $\Big)$.

Show that $A$ is a commutative Frobenius algebra, i.e. a commutative algebra and coalgebra such that

$$(m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta) = \Delta \circ m = (\mathrm{id} \otimes m) \circ (\Delta \otimes \mathrm{id}).$$

**Definition.** Let $x, y$ be two dualizable objects in a symmetric monoidal category $\mathcal{C}$, and $f : x \to y$ be a morphism. Then the dual morphims $f^\vee$ is given by

$$f^\vee : y^\vee \xrightarrow{\mathrm{id}_{y^\vee} \otimes \mathrm{coev}_x} y^\vee \otimes x \otimes x^\vee \xrightarrow{\mathrm{id}_{y^\vee} \otimes f \otimes \mathrm{id}_{x^\vee}} y^\vee \otimes y \otimes x^\vee \xrightarrow{\mathrm{ev}_y \otimes \mathrm{id}_{x^\vee}} x^\vee.$$

(4) *Guided proof of classification theorem*

**Theorem.** *Let $\mathcal{C}$ be a symmetric monoidal category. Then the map*

$$\Phi : TFT^{or}_{1,0}(\mathcal{C}) \to (\mathcal{C}^{dualizable})^{\sim}$$
$$\mathcal{Z} \mapsto \mathcal{Z}(\bullet +)$$

*is an equivalence of groupoids.*

(a) Let $\eta : F \Rightarrow G$ be a symmetric monoidal natural transformation between two symmetric monoidal functors $F, G : \mathcal{C} \to \mathcal{D}$. Show that for any dualizable object $x \in \mathcal{C}$, we have that $\eta(x)$ is invertible and

$$\eta(x^\vee) = (\eta(x)^{-1})^\vee = (\eta(x)^\vee)^{-1}.$$

(b) Conclude that $\mathrm{TFT}^{\mathrm{or}}_{1,0}$ is a groupoid.

(c) Prove that $\Phi$ is faithful, i.e. injective on morphisms.

(d) Prove that $\Phi$ is full, i.e. surjective on morphisms.

(e) Prove that $\Phi$ is essentially surjective, i.e. every object in the target category $(\mathcal{C}^{\text{dual.}})^{\sim}$ is isomorphic to an object in the image of $\Phi$.