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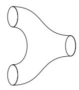
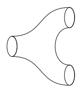
Bordisms and TFTs - Exercise 5

(1) *TFTs valued in* $(\mathit{Vect}_{\Bbbk}, \otimes)$

(a) Let $\mathcal{Z} : \mathrm{Bord}_{n,n-1} \to \mathrm{Vect}_{\Bbbk}$ be an n-dimensional topological field theory. Prove that for every closed $(n\text{-}1)$-manifold Y the vector space $F(Y)$ is finite dimensional. *If you want a hint, see the third page.*

(2) *Algebra and coalgebra structure from a 2-dimensional TFT*

Let $\mathcal{Z} : \mathrm{Bord}_{2,1} \to \mathrm{Vect}_{\Bbbk}$ denote a 2-dimensional topological field theory. Set $A = \mathcal{Z}(S^1)$,

$m := \mathcal{Z}\Big($  $\Big)$ and $\Delta := \mathcal{Z}\Big($  $\Big)$.

(a) Show that m is a unital, associative and commutative product on A.

(b) Show that Δ is a coassociative and cocommutative coproduct

$$\Delta \colon A \longrightarrow A \otimes A.$$

That is, show that Δ satisfies

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta, \qquad (\textit{coassociativity}),$$
$$\beta \circ \Delta = \Delta, \qquad (\textit{cocommutativity}),$$

where β denotes the braiding in $\mathrm{Vect}_{\Bbbk}$. Find a counit ε for A, i.e. a map $\varepsilon : A \to \Bbbk$ such that

$$(\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id} = (\varepsilon \otimes \mathrm{id}) \circ \Delta.$$

Definition. An *isotopy* is a smooth map $F \colon [0,1] \times Y \to Y$ such that $F(t,-) \colon Y \to Y$ is a diffeomorphism for all $t \in [0,1]$. A *pseudoisotopy* is a diffeomorphism $\widetilde{F} : [0,1] \times Y \to [0,1] \times Y$ which preserves the submanifolds $\{0\} \times Y$ and $\{1\} \times Y$. For a manifold Y, the group of diffeomorphisms modulo isotopies is called the *mapping class group* $MCG(Y)$ of Y.

(3) *Isotopy and groupoids*

(a) Let Y be a closed $(n\text{-}1)$-manifold. Show that (pseudo-)isotopic diffeomorphisms of Y give equal bordisms in $\mathrm{Hom}_{\mathrm{Bord}_{n,n-1}}(Y,Y)$. Conclude that there is a homomorphism $MCG(Y) \to \mathrm{Bord}_{n,n-1}(Y,Y)$. Is is injective for $n = 1$ and $Y = * \amalg *$?

(b) Given a category $\mathcal{C}$, one can form the quotient $\pi_0 \mathcal{C} = \mathrm{Ob}\,\mathcal{C}/\sim$, where 2 objects are equivalent if there is a morphism between them (in either direction). What is $\pi_0 \mathrm{Bord}_{n,n-1}$?

(c) Instead, given a category $\mathcal{C}$, one can[1] form a groupoid $|\mathcal{C}|$ by formally "inverting" all morphisms. How is $|\mathrm{Bord}_{n,n-1}|$ related to $\pi_0\mathrm{Bord}_{n,n-1}$? What is $\pi_1|\mathrm{Bord}_{1,0}| = \mathrm{Hom}_{\mathrm{Bord}_{1,0}}(\varnothing, \varnothing)$? Can you recover $\mathrm{Bord}_{1,0}$ from π_0 and π_1? What about for the oriented bordism category?

(d) Try generalizing the situation of (c) to a general symmetric monoidal groupoid. What extra structure/property might be useful?

(e) What is the largest symmetric monoidal subgroupoid in $\mathrm{Bord}_{1,0}$ respectively $\mathrm{Bord}_{2,1}$ containing all objects?

(f) *Hard. Food for thought and further reading.* Is it true that the largest symmetric monoidal subgroupoid in $\mathrm{Bord}_{n,n-1}$ is $(\mathrm{Man}_n, \text{diffeomorphisms})$?

[1] This may require thought about set-theoretic issues. See "groupoid completion" in `https://web.ma.utexas.edu/users/dafr/bordism.pdf`

Hint for Exercise 1: Let $V := \mathcal{Z}(Y)$. Consider the two bordisms obtained from $Y \times [0,1]$ bent to a "cup" or a "cap". Find a composition thereof which is diffeomorphic to id_Y. What does this relation in $\mathrm{Bord}_{n-1,n}$ tell you about the vector space V?