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Bordisms and TFTs - Exercise 4

Definition. An *elementary cobordism* is a manifold with boundary posessing an admissible Morse function f with exactly one critical point p.

(1) *Elementary cobordisms*

(a) Show that an elementary cobordism determines a cobordism.

(b) Find all the elementary unoriented cobordisms for $n = 1, 2$.

(c) Find 3 non-examples of elementary cobordisms.

(d) Decompose Σ_2, i.e. the genus 2 surface, into elementary cobordisms in two ways.

(e) Which connected surfaces can be framed?

(2) *Euler characteristic (mod 2) as a cobordism invariant.*

(a) Let M, N be two n-dimensional compact closed manifolds. Prove the following formulas for the Euler characteristic:

(i) $\chi(M \times N) = \chi(M)\chi(N)$,

(ii) $\chi(\partial M) = (1 - (-1)^{\dim M})\chi(M)$.

(b) Prove that the Euler characteristic modulo 2 is an unoriented cobordism invariant.

Definition. The *braid group* B_n is the group with generators $\sigma_1, \sigma_2, ..., \sigma_{n-1}$ subject to the relations $\sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i$ for all $1 \leqslant i \leqslant n-2$, and $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $i - j \geqslant 2$.

Definition. The *symmetric group* S_n is the group with generators $s_1, s_2, ..., s_{n-1}$ subject to the relations $s_{i+1}s_is_{i+1} = s_is_{i+1}s_i$ for all $1 \leqslant i \leqslant n-2$, $s_is_j = s_js_i$ for $i - j \geqslant 2$ and $s_i^2 = 1$ for all $i \in \{1, ..., n-1\}$.

Remark. Given a family of groups $\{G_n\}_{n\in\mathbb{N}}$ such that $G_0 = \{1\}$ we can build a category G whose objects are the non-negative integers and whose morphisms are given by

$$\mathrm{Hom}_G(m, n) = \begin{cases} \varnothing & \text{if } m \neq n \\ B_n & \text{if } m = n. \end{cases}$$

(3) Show that the below categories are braided monoidal. Which ones are symmetric monoidal?

(a) $(\mathrm{Vect}_{\Bbbk}, \oplus)$, with braiding the flip map $\tau : V \oplus W \to W \oplus V$.

(b) $(\mathrm{Vect}_{\Bbbk}, \otimes)$, with braiding the flip map $\tau : V \otimes W \to W \otimes V$.

(c) $B = \big(\{B_n\}_{n\in\mathbb{N}}, \otimes\big)$, where $\otimes : B \times B \to B$ is defined by addition on objects and group homomorphisms $\rho_{m,n} : B_n \times B_m \to B_{n+m}$ given by $\rho_{m,n}(\sigma_i, \sigma_j) = \sigma_i\sigma_{m+j}$ for all $1 \leqslant i \leqslant m-1$, $1 \leqslant j \leqslant n-1$. The braiding $\beta = \beta_{m,n} : m+n \to n+m$ is defined as follows: set $\gamma = \sigma_n\sigma_{n-1}\dots\sigma_1 \in B_{n+1}$ and $\gamma^{(p)} = 1_p + \gamma + 1_{m-p-1} \in B_{m+n}$ for $p \in \{0, 1, ..., m-1\}$. Then $\beta_{m,n} := \gamma^{(0)}\gamma^{(1)}\dots\gamma^{(m-1)}$.
Hint: Finding a pictorial description of the above can be helpful.

(d) $S = \big(\{S_n\}_n, \otimes\big)$ with the analogous braided monoidal structure as in (c).