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Bordisms and TFTs - Exercise 7

(1) *Equivalent definitions of Frobenius algebras*

(a) Let A be a Φ-Frobenius algebra. Show that defining

$$\kappa : A \otimes A \to \Bbbk$$

as $\kappa(a, b) := \Phi_b(a)$ gives A the structure of a κ-Frobenius algebra.

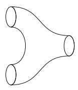
(b) Conversely, let A be a κ-Frobenius algebra. Show that defining

$$\Phi : {}_A A \to (A_A)^*$$

as $\Phi(b)(a) := \kappa(a, b)$ gives A the structure of a Φ-Frobenius algebra.

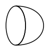
(2) *TFT from a group algebra*

(a) Let G be a finite, abelian group, and let $\Bbbk[G]$ denote the group algebra of G. Set $\mathcal{Z}(S^1) = \Bbbk[G]$, and let the multiplication and counit of $\Bbbk[G]$ be given by the linear extensions of

$m := \mathcal{Z}\Big($  $\Big)$: $\quad \Bbbk[G] \otimes \Bbbk[G] \to \Bbbk[G]$

$$g \otimes h \mapsto gh$$

and

$\mathcal{Z}($  $)$: $\quad \Bbbk[G] \to \Bbbk$

$$g \mapsto \delta_{g,e}.$$

Show that this assignment defines an oriented topological field theory

$$\mathcal{Z} : \mathrm{Bord}_{2,1}^{\mathrm{or}} \to \mathrm{Vect}_{\Bbbk}.$$

(b) Using Exercise 1, what is the corresponding Φ-Frobenius structure on $\Bbbk[G]$?

(c) Compute $\Delta : \Bbbk[G] \to \Bbbk[G] \otimes \Bbbk[G]$. *Hint:* Use the basis of $\Bbbk[G]$.

(d) Compute the value of the TFT on the genus g surface with 2 disks removed, i.e. compute $\mathcal{Z}(\Sigma_g \backslash D \sqcup D)$.

(3) *Examples of Frobenius algebras*

(a) Show that $\mathbb{C}$ is a Frobenius algebra over $\mathbb{R}$ with Frobenius form induced by

$$\varepsilon : \mathbb{C} \to \mathbb{R}$$
$$a + bi \mapsto a.$$

Could we have chosen a different map $\mathbb{C} \to \mathbb{R}$?

(b) Let G be a finite group of order n. A *class function* on G is a function $G \to \mathbb{C}$ which is constant on each conjugacy class[1]. The class functions of G form a ring $R(G)$. Show that the bilinear pairing

$$\kappa(\phi, \psi) := \frac{1}{n} \sum_{t \in G} \phi(t)\psi(t^{-1})$$

gives $R(G)$ the structure of a κ-Frobenius algebra.

(c) Let X be a compact oriented manifold of dimension n and let $H^*(X) = \oplus_{i=0}^{n} H^i(X)$ denote the de Rham cohomology, which is a ring under the wedge product. Show that the pairing

$$\int : \quad H^*(X) \otimes H^*(X) \to \mathbb{R}$$

$$(\alpha, \beta) \mapsto \int_X \alpha \wedge \beta$$

gives $H^*(X)$ the structure of a κ-Frobenius algebra over $\mathbb{R}$.

(d) Look up additional examples of Frobenius algebras.

[1] Characters, i.e. traces of representations, are examples of class functions.