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Bordisms and TFTs - Exercise 8

Remark. Details and full solutions for the first three exercises can be found at the two links (upside down) on the last page.

(1) *Class functions as Frobenius algebra*

(a) Let G be a finite group of order n. A *class function* on G is a function $G \to \Bbbk$ which is constant on each conjugacy class. The class functions of G form a ring $\mathrm{Map}(G, \Bbbk)^G$ under the convolution product, i.e.

$$\phi * \psi(x) := \sum_{x_1 x_2 = x} \phi(x_1)\psi(x_2).$$

Show that the bilinear pairing

$$\kappa(\phi, \psi) := \frac{1}{n} \sum_{t \in G} \phi(t)\psi(t^{-1})$$

gives $\mathrm{Map}(G, \Bbbk)^G$ the structure of a κ-Frobenius algebra.

(b) Show that under the identification

$$\begin{aligned} \mathrm{Map}(G, \Bbbk)^G &\to \Bbbk[G] \\ f &\mapsto \sum_{g \in G} f(g) g, \end{aligned}$$

$\mathrm{Map}(G, \Bbbk)^G$ bijectively corresponds to the center $Z(\Bbbk[G])$ of the group algebra.

(2) *Principal G-bundles*

(a) Let G denote a topological group, and let $\mathrm{Prin}_G(X)$ denote the isomorphism classes of principal G-bundles over the space X. Show that

$$\mathrm{Prin}_G(S^1) \cong \mathrm{Hom}(\mathbb{Z}, G)/\sim \; = G/\sim,$$

where the equivalence relation $\sim$ is generated by conjugation in G.

(b) Show that for any surface X we have a bijection

$$\mathrm{Hom}(\pi_1(X), G)/G \xrightarrow{\sim} \mathrm{Prin}_G(X)$$

Remark. Note that if M is a bordism in $\mathrm{Bord}_2^{\mathrm{or}}$ with boundary $\partial M = \Sigma_1 \sqcup \Sigma_2$, restricting a principal G-bundle on M to either of the boundary components Σ_i, $i \in \{1, 2\}$, yields a principal G-bundle $P\big|_{\Sigma_i} \to \Sigma_i$.

Remark. If $P \to M$ is a principal G-bundle, then we denote by $\mathrm{Aut}(P)$ the G-equivariant homeomorphisms $P \to P$ that cover the identity of M.

(3) *Dijkgraaf-Witten theory*

Dijkgraaf-Witten theory is a oriented 2-TFT $\mathcal{Z}$ constructed by sending objects M to $\mathcal{Z}(M) := \mathrm{Map}(\mathrm{Prin}_G(M), \Bbbk)$. For Σ a oriented 2-dimensional bordism from M_1 to M_2 the assignment is

$$\mathcal{Z}(\Sigma): \quad \mathcal{Z}(M_1) \longrightarrow \mathcal{Z}(M_2)$$

$$\mathcal{Z}(\Sigma)(f)(p_2) = \sum_{p_1 \in \mathrm{Prin}_G(M_1)} \sum_{p \in C_\Sigma(p_1, p_2)} f(p_1) \frac{\#\mathrm{Aut}(p_2)}{\#\mathrm{Aut}(p)},$$

where $C_\Sigma(p_1, p_2) = \{p \in \mathrm{Prin}_G(\Sigma) | \, p_{|M_i} = p_i, i \in \{1, 2\}\}$.

(a) Compute $\mathcal{Z}(\Sigma)$ for Σ the pair of pants.

(b) Compute $\mathcal{Z}(\Sigma_g \backslash D \sqcup D)$. Compare the result with what was obtained in Exercise 2 (d) on Sheet 7.

(c) Compare the advantages and disadvantages of this approach to the the one assigning the group algebra to the circle and using the classification.

Details for Exercises 1-3 can be found at: `https://upennig.weebly.com/uploads/7/4/0/3/7403187/2d-tqft.pdf https://math.berkeley.edu/~qchu/TQFT.pdf`