

### BORDISMS AND TFTS - EXERCISE 11

(1) *(co)Commutativity of Hopf algebras*

- (a) Let  $H$  be a Hopf algebra in the category  $\text{Vect}_{\mathbb{k}}$ . Prove that the relation  $S^2 = \text{id}_H$  is equivalent to

The diagram shows a vertical line that loops back on itself, forming a figure-eight shape. A small circle with the letter 's' is attached to the right side of the lower loop. This is set equal to two separate vertical lines, each with a small circle in the middle.

- (b) Let  $H$  be a finite dimensional Hopf algebra (again in  $\text{Vect}_{\mathbb{k}}$ ). Assume that the antipode  $S$  has odd order (i.e. the smallest positive  $n$  such that  $S^n = \text{id}_H$  is odd). Prove that  $H$  is commutative and cocommutative. Conclude that  $S = \text{id}_H$  in this situation.
- (c) Can you find an example of such a Hopf algebra?
- (2) *Reidermeister moves and the Jones polynomial*

- (a) Show that the below move is a consequence of the three Reidemeister moves (given in the student presentation on Tuesday 22.06).

The diagram shows a vertical line on the left and a loop on the right. A double-headed arrow points from the line to the loop, indicating an equivalence between the two configurations.

- (b) Compute the Jones polynomial for the positive Hopf link  $H^+$  in two different ways: First by the Kaufmann bracket and then using the skein relations.
- (c) Compute the Jones polynomial of the negative Hopf link  $H^-$ . Do you see any relation between this result and that of the previous part-exercise?

The diagram shows two links. On the left is the positive Hopf link  $H^+$ , consisting of two linked circles with arrows indicating a positive crossing. On the right is the negative Hopf link  $H^-$ , consisting of two linked circles with arrows indicating a negative crossing. A comma separates the two diagrams.

**Definition.** A Lie algebra is a vector space  $\mathfrak{g}$  together with an alternating bilinear map  $[-, -] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ , called the *Lie bracket*, satisfying the Jacobi identity:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \quad \text{for all } x, y, z \in \mathfrak{g}.$$

(3) *Lie algebras*

(a) Let  $\mathfrak{sl}_2(\mathbb{C})$  be the vector space of all two-by-two complex matrices with zero trace. Prove that the Lie bracket given by the commutator (i.e.  $[A, B] = AB - BA$ ) gives  $\mathfrak{sl}_2(\mathbb{C})$  the structure of a Lie algebra.

(b) Show that any associative  $\mathbb{k}$ -algebra  $A$  inherits a structure of a Lie algebra by using the commutator, i.e.  $[a, b] = ab - ba$  for all  $a, b \in A$ .

(4) *Food for thought (Hard!)*

(a) Consider the inclusion  $\iota : (X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ . Prove surjectivity of the induced morphism  $\iota_* : N_n(X \setminus Z, A \setminus Z) \rightarrow N_n(X, A)$  for  $\overline{Z} \subseteq \text{int}(A)$ . Can you also prove injectivity of  $\iota_*$ ? (This is Exercise 3 in the notes from the student presentation on Wednesday 23.06. See there for more details.)