Summer semester 2021

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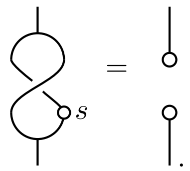
Bordisms and TFTs - Exercise 11

(1) *(co)Commutativity of Hopf algebras*

(a) Let H be a Hopf algebra in the category $\mathrm{Vect}_{\Bbbk}$. Prove that the relation $S^2 = \mathrm{id}_H$ is equivalent to

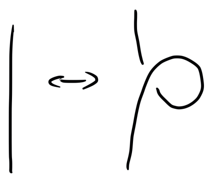


(b) Let H be a finite dimensional Hopf algebra (again in $\mathrm{Vect}_{\Bbbk}$). Assume that the antipode S has odd order (i.e. the smallest positive n such that $S^n = \mathrm{id}_H$ is odd). Prove that H is commutative and cocommutative. Conclude that $S = \mathrm{id}_H$ in this situation.

(c) Can you find an example of such a Hopf algebra?

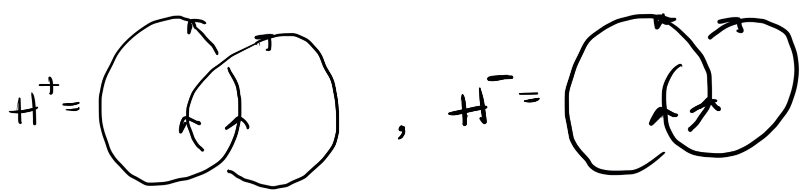
(2) *Reidermeister moves and the Jones polynomial*

(a) Show that the below move is a consequence of the three Reidermeister moves (given in the student presentation on Tuesday 22.06).



(b) Compute the Jones polynomial for the positive Hopf link H^+ in two different ways: First by the Kaufmann bracket and then using the skein relations.

(c) Compute the Jones polynomial of the negative Hopf link H^-. Do you see any relation between this result and that of the previous part-exercise?



Definition. A Lie algebra is a vector space $\mathfrak{g}$ together with an alternating bilinear map $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the *Lie bracket*, satisfying the Jacobi identity:

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \qquad \text{for all } x,y,z \in \mathfrak{g}.$$

(3) *Lie algebras*

(a) Let $\mathfrak{sl}_2(\mathbb{C})$ be the vector space of all two-by-two complex matrices with zero trace. Prove that the Lie bracket given by the commutator (i.e. $[A,B] = AB - BA$) gives $\mathfrak{sl}_2(\mathbb{C})$ the structure of a Lie algebra.

(b) Show that any associative $\Bbbk$-algebra A inherits a structure of a Lie algebra by using the commutator, i.e. $[a,b] = ab - ba$ for all $a,b \in A$.

(4) *Food for thought (Hard!)*

(a) Consider the inclusion $\iota : (X \backslash Z, A \backslash Z) \hookrightarrow (X, A)$. Prove surjectivity of the induced morphism $\iota_* : N_n(X \backslash Z, A \backslash Z) \to N_n(X, A)$ for $\overline{Z} \subseteq \operatorname{int}(A)$. Can you also prove injectivity of ι_*? (This is Exercise 3 in the notes from the student presentation on Wednesday 23.06. See there for more details.)