Summer semester 2021

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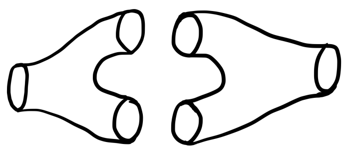
Bordisms and TFTs - Exercise 1

Note: Remarks in parentheses at the beginning of an exercise refers to necessary prerequisites. There is an invite-link to the Rocketchat-channel of the course available on the Moodle-page, please use it to join the group!

(1) Show that the disjoint union induces an abelian group structure on Ω_n.

(2) Transitivity of the cobordism relation for surfaces:

(a) View the 2-sphere with 3 open disks removed as a cobordism in 2 ways: First as a *pair of pants*, i.e. a bordism $\Sigma_{2,1}$ from $S^1 \sqcup S^1$ to S^1, and secondly as a *pair of copants*, i.e. a bordism $\Sigma_{1,2}$ from S^1 to $S^1 \sqcup S^1$. Show that $\Sigma_{1,2} \underset{S^1 \sqcup S^1}{\sqcup} \Sigma_{2,1}$ can be viewed as a bordism from S^1 to S^1.



(b) Argue that a similar thing holds for all surfaces, i.e. that bordism of surfaces is a transitive relation.

(3) Recollection from Lecture 2:

(a) Work through the argument in detail showing that Σ_g is cobordant to the empty set.

(b) Recall that the disjoint union is cobordant to the connected sum. Work through the details for an example that is different from what was shown in the lecture.

(c) Conclude that $\Omega_2^{\mathrm{or}} = 0$. (Here you may omit details about orientations of the 3-dimensional cobordisms.)

(4) Computation of Ω_2^{unor}:

(a) Show that the Klein bottle K is cobordant to the empty set.

(b) (Poincaré duality and Euler characteristic) Show that $\mathbb{RP}^2$ is non-zero in Ω_2^{unor}, i.e. that there is no compact 3-manifold X with boundary $\partial X = \mathbb{RP}^2$.
Hint: Consider the double $D = X \cup_{\mathbb{RP}^2} X$. What does Poincaré duality imply about the Euler characteristic of 3-dimensional closed manifolds?

(c) Conclude from the above that $\Omega_2^{\mathrm{unor}} = \mathbb{Z}/2\mathbb{Z}$.