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Bordisms and TFTs - Exercise 3

(1) *Morse functions*

(a) Consider the height function from the torus to $\mathbb{R}$ (first Example in Lecture 5). Mark out the critical points again and find their index.

(b) For the following functions, what are the critical points? Are they degenerate or non-degenerate? Is it a Morse function? If not, perturb it so that it becomes Morse.

(i) $f : \mathbb{R} \to \mathbb{R}$, $(x, y) \mapsto x^3 - 3xy^2$.

(ii) $g : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto x^2y^2$.

(c) Show that if $f : M \to \mathbb{R}$ and $g : N \to \mathbb{R}$ are Morse functions, then $f + g : M \times N \to \mathbb{R}$ is also a Morse function, and the critical points are pairs of critical points of f and g. Try to visualize this for $M = N = S^1$ and $f\colon S^1 \subset \mathbb{R}^2 \to \mathbb{R}$ the projection onto the first coordinate.

Definition. Let M be a compact 2-manifold. A *handle decomposition* of M is a finite sequence of manifolds

$$\varnothing = W_{-1} \subseteq W_0 \subseteq W_1 \subseteq W_2 = M$$

such that each W_i is obtained from W_{i-1} by attaching i-handles.

(2) *Handle decompositions*

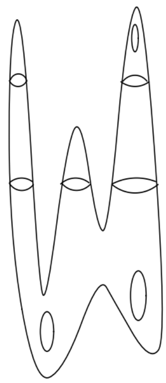
(a) Find two different handle decompositions of S^2.

(b) Find a handle decomposition of $\mathbb{RP}^2$.

(c) Find a handle decomposition of the Klein bottle.

(d) Explain why for any non-empty closed connected surface we can start a handle decomposition with a single 0-handle. *Hint: The key argument was mentioned in Lecture 5 as* "handle cancellation"*.*

(e) Using the idea of handle cancellation, bring the below surface into *normal form*, i.e. such that read from bottom to top the index of the critical points are non-decreasing.



(3) *Classification of closed 1-manifolds*

(a) Try to prove the below (familiar) theorem using Morse theory and/or read through the proof (of Theorem 15) in `https://www.math.csi.cuny.edu/~abhijit/papers/classification.pdf`.

Theorem. *Any closed 1-manifold is homeomorphic to S^1.*

Can you extend the argument to replace "homeomorphic" by "diffeomorphic"?