### Bordisms and TFTs - Exercise 10

**Definition.** A *ribbon category* is a monoidal category $C$ that has right duals, a braiding and a twist, i.e. a natural family of isomorphisms $\theta = \{\theta_x : x \to x\}_{x \in \text{ob}(C)}$ such that for any two objects $x, y$ in $C$ we have

$$\theta_{x \otimes y} = \beta_{y,x} \circ \beta_{x,y} \circ (\theta_x \otimes \theta_y),$$

and the braiding, duality and twist are compatible in the following way:

$$\theta_x \circ \text{id}_x \_ \theta_x \circ \text{coev}_x = (\text{id}_x \otimes \theta_x) \circ \text{coev}_x.$$

1. **Ribbon categories**

   Prove that the definition of a ribbon category given in the lecture is equivalent to the above definition.

2. **Sweedler’s Hopf algebra**

   (a) Consider the $\mathbb{C}$-algebra $H$ generated by two elements $C$ and $X$ subject to the relations

   $$C^2 = 1, \quad X^2 = 0 \quad \text{and} \quad CX + XC = 0.$$

   The comultiplication, counit and antipode are defined by

   $$\Delta(C) = C \otimes C, \quad \Delta(X) = 1 \otimes X + X \otimes C,$$

   $$\varepsilon(C) = 1, \quad \varepsilon(X) = 0, \quad S(C) = C \quad \text{and} \quad S(X) = CX.$$

   Prove that this indeed defines a Hopf algebra.

   (b) Is this Hopf algebra (co)commutative?

3. **The quantum enveloping algebra of $\mathfrak{sl}(2)$**

   Let $q$ be an invertible element of $\mathbb{k}$ different from 1 and $-1$ so that the fraction $\frac{1}{q - q^{-1}}$ is well-defined. Define $U_q = U_q(\mathfrak{sl}_2)$ as the algebra generated by the four variables $E, F, K, K^{-1}$ subject to the relations

   $$KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2} F \quad \text{and} \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

   Let the comultiplication, counit and antipode be defined by

   $$\Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1, \quad \Delta(K) = K \otimes K,$$

   $$\Delta(K^{-1}) = K^{-1} \otimes K^{-1}, \quad \varepsilon(E) = 0 = \varepsilon(F), \quad \varepsilon(K) = 1 = \varepsilon(K^{-1}),$$

   $$S(E) = -EK^{-1}, \quad S(F) = -KF, \quad S(K) = K^{-1} \quad \text{and} \quad S(K^{-1}) = K.$$

**Footnote:** This is the definition given in Kassel-Rosso-Turaev (p.60), and it also appears frequently in the literature.
Prove that this defines a Hopf algebra.

Remark: If you want a readable intro to quantum groups you can e.g. have a look at https://www.ams.org/notices/200601/what-is.pdf