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Bordisms and TFTs - Exercise 10

Definition. A *ribbon category*[1] is a monoidal category $\mathcal{C}$ that has right duals, a braiding and a twist, i.e. a natural family of isomorphisms $\theta = \{\theta_x : x \to x\}_{x \in \mathrm{ob}(\mathcal{C})}$ such that for any two objects x, y in $\mathcal{C}$ we have

$$\theta_{x \otimes y} = \beta_{y,x} \circ \beta_{x,y} \circ (\theta_x \otimes \theta_y),$$

and the braiding, duality and twist are compatible in the following way:

$$(\theta_x \otimes \mathrm{id}_{x^\vee}) \circ \mathrm{coev}_x = (\mathrm{id}_x \otimes \theta_{x^\vee}) \circ \mathrm{coev}_x.$$

(1) *Ribbon categories*

Prove that the definition of a ribbon category given in the lecture is equivalent to the above definition.

(2) *Sweedler's Hopf algebra*

(a) Consider the $\mathbb{C}$-algebra H generated by two elements C and X subject to the relations

$$C^2 = 1, \quad X^2 = 0 \quad \text{and} \quad CX + XC = 0.$$

The comultiplication, counit and antipode are defined by

$$\Delta(C) = C \otimes C, \quad \Delta(X) = 1 \otimes X + X \otimes C,$$
$$\varepsilon(C) = 1, \quad \varepsilon(X) = 0, \quad S(C) = C \quad \text{and} \quad S(X) = CX.$$

Prove that this indeed defines a Hopf algebra.

(b) Is this Hopf algebra (co)commutative?

(3) *The quantum enveloping algebra of* $\mathfrak{sl}(2)$

Let q be an invertible element of $\Bbbk$ different from 1 and -1 so that the fraction $\frac{1}{q-q^{-1}}$ is well-defined. Define $U_q = U_q(\mathfrak{sl}_2)$ as the algebra generated by the four variables E, F, K, K^{-1} subject to the relations

$$KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F \quad \text{and} \quad [\mathrm{E}, \mathrm{F}] = \frac{\mathrm{K} - \mathrm{K}^{-1}}{\mathrm{q} - \mathrm{q}^{-1}}.$$

Let the comultiplication, counit and antipode be defined by

$$\Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1, \quad \Delta(K) = K \otimes K,$$
$$\Delta(K^{-1}) = K^{-1} \otimes K^{-1}, \quad \varepsilon(E) = 0 = \varepsilon(F), \quad \varepsilon(K) = 1 = \varepsilon(K^{-1}),$$
$$S(E) = -EK^{-1}, \quad S(F) = -KF, \quad S(K) = K^{-1} \quad \text{and} \quad S(K^{-1}) = K.$$

[1]This is the definition given in Kassel-Rosso-Turaev (p.60), and it also appears frequently in the literature.

Prove that this defines a Hopf algebra.
Remark: If you want a readable intro to quantum groups you can e.g. have a look at `https://www.ams.org/notices/200601/what-is.pdf`.