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Bordisms and TFTs - Exercise 12

(1) *Universal and quantum enveloping algebra of* $\mathfrak{sl}_2(\mathbb{C})$

(a) Using the basis h, e and f given below, find the relations that defines the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$[1].

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

(b) Write down a presentation of the universal enveloping algebra $U := U(\mathfrak{sl}_2(\mathbb{C}))$.

(c) Using the below Proposition, prove that we have the following relationship

$$U'_{q=1} \cong U[K]/(K^2 - 1)$$

between the universal and quantum enveloping algebras of $\mathfrak{sl}_2(\mathbb{C})$.

Proposition. *The algebra* $U_q := U_q(\mathfrak{sl}_2(\mathbb{C}))$[2] *is isomorphic to the algebra* U'_q *generated by the five variables* E, F, K, K^{-1}, L *and the relations*

$$KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F \quad [E,F] = L,$$

$$(q - q^{-1})L = K - K^{-1}, \quad [L,E] = q(EK + K^{-1}E)$$

and $[L,F] = -q^{-1}(FK + K^{-1}F)$,

where the parameter q *is allowed to take any value.*

Definition. Any closed subgroup of $GL(n,\mathbb{C})$ is a Lie group, and Lie groups of this type are called *matrix Lie groups.*

Fact. *For each element of the tangent space at* $I \in G$*, one can construct a flow and hence a corresponding one-parameter subgroup* $A(t)$*. In the situation where* G *is a matrix Lie group, every one-parameter subgroup has the form* $A(t) = e^{tX}$ *for some matrix* $X = A'(0)$*. From this we get the following result for matrix groups*

$$X \in \mathfrak{g} \iff e^{tX} \in G \quad \forall\, t \in \mathbb{R}.$$

(2) *(Some linear algebra) Exponential map between matrix Lie algebras and their corresponding Lie groups*

(a) For any matrix $X \in M_n(\mathbb{C})$ the exponential map is given by

$$e^X = \sum_{k=0}^{\infty} \frac{1}{k!} X^k.$$

[1] This Lie algebra was defined in class as $\mathfrak{sl}(\mathbb{C}^2)$ or in Exercise 3 (a) on Sheet 11 as the 2×2 complex matrices with trace 0.

[2] This algebra was defined in Exercise 3 on Sheet 10.

Prove that for two matrices $A, B \in M_n(\mathbb{C})$ that commute, i.e. $[A, B] = 0$, we have $e^A e^B = e^{A+B}$.

(b) Using the above fact, show that the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ corresponding to the Lie group $GL(n, \mathbb{C})$ is given by all complex $n \times n$ matrices, i.e. $\mathfrak{gl}_n(\mathbb{C}) = M_n(\mathbb{C})$. This proves that $\mathfrak{gl}(\mathbb{C}^n) = \mathrm{End}(\mathbb{C}^n)$ as we defined in the lecture.

(c) Using the relation $\det(e^A) = e^{\mathrm{tr}(A)}$, find the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ corresponding to the Lie group $SL(n, \mathbb{C})$.

(d) Can you find other examples of matrix Lie groups and their corresponding matrix Lie algebras?