BORDISMS AND TFTs - EXERCISE 12

(1) Universal and quantum enveloping algebra of $\mathfrak{sl}_2(\mathbb{C})$

(a) Using the basis $h, e$ and $f$ given below, find the relations that defines the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$:

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

(b) Write down a presentation of the universal enveloping algebra $U := U(\mathfrak{sl}_2(\mathbb{C}))$.

(c) Using the below Proposition, prove that we have the following relationship between the universal and quantum enveloping algebras of $\mathfrak{sl}_2(\mathbb{C})$.

Proposition. The algebra $U_q := U_q(\mathfrak{sl}_2(\mathbb{C}))$ is isomorphic to the algebra $U'_q$ generated by the five variables $E, F, K, K^{-1}, L$ and the relations

$$KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F, \quad [E,F] = L,$$

$$(q - q^{-1})L = K - K^{-1}, \quad [L,E] = q(EK + K^{-1}E)$$

and $$[L,F] = -q^{-1}(FK + K^{-1}F),$$

where the parameter $q$ is allowed to take any value.

Definition. Any closed subgroup of $GL(n, \mathbb{C})$ is a Lie group, and Lie groups of this type are called matrix Lie groups.

Fact. For each element of the tangent space at $I \in G$, one can construct a flow and hence a corresponding one-parameter subgroup $A(t)$. In the situation where $G$ is a matrix Lie group, every one-parameter subgroup has the form $A(t) = e^{tX}$ for some matrix $X = A'(0)$. From this we get the following result for matrix groups

$$X \in \mathfrak{g} \iff e^{tX} \in G \quad \forall t \in \mathbb{R}.$$

(2) (Some linear algebra) Exponential map between matrix Lie algebras and their corresponding Lie groups

(a) For any matrix $X \in M_n(\mathbb{C})$ the exponential map is given by

$$e^X = \sum_{k=0}^{\infty} \frac{1}{k!} X^k.$$

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1This Lie algebra was defined in class as $\mathfrak{sl}(\mathbb{C}^2)$ or in Exercise 3 (a) on Sheet 11 as the $2 \times 2$ complex matrices with trace 0.

2This algebra was defined in Exercise 3 on Sheet 10.
Prove that for two matrices $A, B \in M_n(\mathbb{C})$ that commute, i.e. $[A, B] = 0$, we have $e^A e^B = e^{A+B}$.

(b) Using the above fact, show that the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ corresponding to the Lie group $GL(n, \mathbb{C})$ is given by all complex $n \times n$ matrices, i.e. $\mathfrak{gl}_n(\mathbb{C}) = M_n(\mathbb{C})$. This proves that $\mathfrak{gl}(\mathbb{C}^n) = \text{End}(\mathbb{C}^n)$ as we defined in the lecture.

(c) Using the relation $\det(e^A) = e^{\text{tr}(A)}$, find the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ corresponding to the Lie group $SL(n, \mathbb{C})$.

(d) Can you find other examples of matrix Lie groups and their corresponding matrix Lie algebras?