Summer semester 2021

Claudia Scheimbauer
Eilind Karlsson

Bordisms and TFTs - Exercise 2

(1) *Framings*

(a) Find at least 3 different framings on the strip $(0,1) \times (0,1)$, restricting to the below framings in a neighborhood of $\{0\} \times (0,1)$ and $\{1\} \times (0,1)$.

(b) Which ones determine the same (which) orientation? How would you define "homotopy of framing"?

(c) Can S^2 be framed?

Definition. Let B^n denote the n-dimensional ball as a manifold with boundary and S^n the n-dimensional sphere.

Given a 2-dimensional manifold M, we *attach a j-handle* $H^j \coloneqq B^j \times B^{2-j}$, for $j \in \{0,1,2\}$ via and a smooth embedding $f : S^{j-1} \times B^{2-j} \hookrightarrow \partial M$ as follows:

$$M \cup_f H^j \coloneqq \big(M \sqcup (B^j \times B^{2-j})\big)/\sim$$

where for $(p,x) \in S^{j-1} \times B^{2-j} \subset B^j \times B^{2-j}$, we set $f(p,x) \sim (p,x)$.

(2) *Attaching handles*

(a) Find a smooth structure on $M \cup_f H^j$.

(b) Which surface do you obtain from attaching a 1-handle to a disk?

(c) Which surface do you obtain from attaching two 1-handles to a disk, i.e. from attaching an additional 1-handle to the surface obtained in part a?

(d) Build the torus by successively attaching handles to a disk.

(3) *Properties of the connected sum of manifolds*

(a) Given n-manifolds M, M', and M'', show that the connected sum satisfies the following properties.

(i) $M\#S^n \cong M$, *(neutral element)*

(ii) $M\#M' \cong M'\#M$, and *(commutativity)*

(iii) $(M\#M')\#M'' \cong M\#(M'\#M'')$. *(associativity)*

(b) If M and M' are smooth n-manifolds, construct a smooth structure on the connected sum $M \# M'$. Note that this is not unique, but defines a well-defined diffeomorphism class. You may like to read more details using isotopies in Chapter 8, Section 2 in Hirsch, Differential Topology[1].

(4) Below is a list of several proofs of the classification theorem of 1-dimensional manifolds, using different tools. Read through one (or several) of them.

- `https://pnp.mathematik.uni-stuttgart.de/igt/eiserm/lehre/2014/Topologie/Gale%20-%201-manifolds.pdf`
- `http://www.math.boun.edu.tr/instructors/wdgillam/1manifolds.pdf`
- Appendix of `https://www.maths.ed.ac.uk/~v1ranick/papers/milnortop.pdf`, starting at p.55.

[1]Can e.g. be accessed at `https://www.researchgate.net/publication/268035774_Differential_Topology`.