Combinatorics of Coxeter Groups
Reading assignment and Exercises for June 1st

Read section 8.1 to familiarize yourself with a new Coxeter group, S_n^B. I recommend that you ignore formulas (8.1) and (8.3), and instead use (8.2) as definition of inv_B.

Here is a list of exercises to help you understand the group S_n^B (and Coxeter groups in general) better. If you want more exercises, then Exercises 8.1–8.6 of in the book are worth an attempt.

Exercise 1

Describe $T_R(u)$ for $u \in S_n^B$ (Hint: Propositions 8.1.5 and 8.1.6). Give a computable criterion for the left weak order relation $u \leq_L v$ where $u, v \in S_n^B$.

Exercise 2

What is the window notation of the longest element $w_0 \in S_n^B$? Describe the automorphism $w \mapsto w_0 w w_0$ of S_n^B.

Exercise 3

Consider the following elements in S_3^B:

$$x = [-2, 3, -1], \quad u = [3, 2, 1], \quad v = [-1, 3, 2], \quad w = [2, 3, -1].$$

(a) Compute the lengths of these four elements in S_n^B.

(b) Show that $\ell(x) > \ell(u)$ but $x \not\geq u$ in the Bruhat order. Find $i \in \{0, 1, 2\}$ such that $x^J \not\geq u^J$, where $J = S \setminus \{s_i^B\}$.

(c) It is given (and easy to see) that $v = s_0^B s_2^B$ is a reduced expression. Find a reduced expression of x that ends with $s_0^B s_2^B$. Hint: Find a reduced expression for xv^{-1}, then concatenate it with the reduced expression $v = s_0^B s_2^B$ to get an expression for x. Conclude that this expression must be reduced by either comparing lengths, or by using your criterion from Exercise 1 to show $v <_L x$.

(d) It is given that $w \leq x$ (the computation with Theorem 8.1.8 is a bit tedious). Take the reduced expression you got for x in the previous exercise, and find a subexpression that is a reduced expression for w. Show that $w \not\leq_L x$ using your criterion from Exercise 1 or a direct argument.