summer semester 2020

Claudia Scheimbauer
Tashi Walde

Advanced Topics in Algebraic Topology — Exercise Sheet 1

Exercise class: Friday, 8th May; 10-12

Website with further material, including exercise sheets:
https://www.groups.ma.tum.de/algebra/scheimbauer/advanced-topics-in-algebraic-topology/

Recall the definition of Ω^*:

- As a vector space over $\mathbb{R}$ it has a basis given by monomials
$$\mathrm{d}x_I := \mathrm{d}x_{i_1} \cdots \mathrm{d}x_{i_p}$$
for $I = (i_1, \dots, i_p)$ with $1 \leq i_1 < \cdots < i_p \leq n$.
- It is an algebra with multiplication $\wedge \colon \Omega^* \otimes \Omega^* \to \Omega^*$ given on basis vectors by:
$$\mathrm{d}x_{i_1} \dots \mathrm{d}x_{i_p} \wedge \mathrm{d}x_{i_{p+1}} \cdots \mathrm{d}x_{i_{p+q}} := \begin{cases} 0 & \text{if } i_k = i_{p+l} \text{ for some } 1 \leq k, l \leq n \\ \operatorname{sgn}(\sigma) \cdot \mathrm{d}x_{\sigma(i_1)} \cdots \mathrm{d}x_{\sigma(i_{p+q})} & \text{otherwise,} \end{cases}$$
where in the second case σ is the unique permutation of the set $\{i_1, \dots, i_{p+q}\}$ such that $\sigma(i_1) < \cdots < \sigma(i_{p+q})$.

Exercise 1. *Let $(A, \wedge)$ be any algebra over $\mathbb{R}$ with chosen elements $d_1, \dots, d_n \in A$ satisfying*
$$d_i \wedge d_j = -d_j \wedge d_i \qquad \text{for all } i, j \tag{1}$$
Show that there is an unique $\mathbb{R}$-algebra morphism $\Omega^* \to A$ which sends $\mathrm{d}x_i$ to d_i for all i.

Remark. *This is what it means to say that Ω^* is presented as an $\mathbb{R}$-algebra by the **generators** $\mathrm{d}x_i$ and the **relations** (1).*

Exercise 2. *Please (re)familiarize yourselves with the notion of a smooth manifold. Determine whether the following topological spaces (with the subspace topology induced from $\mathbb{R}^2$) are smooth manifolds. (warning: trick question)*

(1) The circle
$$\mathrm{S}^1 := \{z \in \mathbb{C} \,:\, |z| = 1\} \subset \mathbb{C} \cong \mathbb{R}^2;$$

(2) the coordinate axes
$$K := \{(x_1, x_2) \in \mathbb{R}^2 \,:\, x_1 x_2 = 0\};$$

(3) the subsets
$$U := \{(x_1, x_2) \in \mathbb{R}^2 \,:\, x_1 < x_2\},$$
$$\overline{U} := \{(x_1, x_2) \in \mathbb{R}^2 \,:\, x_1 \leq x_2\}$$
of (strictly) above-diagonal points in $\mathbb{R}^2$;

(4) the intersections $K \cap \mathrm{S}^1$, $K \cap U$ and $K \cap \overline{U}$.

Recall the definition of the de Rham algebra $\Omega^*(\mathbb{R}^n) \coloneqq \mathrm{C}^\infty(\mathbb{R}^n) \otimes_{\mathbb{R}} \Omega^*$ and of the de Rham differential $\Omega^*(\mathbb{R}^n) \to \Omega^{*+1}(\mathbb{R}^n)$ given on elementary tensors by

$$f_I \otimes \mathrm{d}x_I \longmapsto \sum_{i=1}^{n} \frac{\partial f_I}{\partial x_i} \otimes \mathrm{d}x_i \wedge \mathrm{d}x_I$$

and extended $\mathbb{R}$-linearly (note that we would typically omit the $\otimes$ symbol).

Exercise 3. *Show that* $(\Omega^*(\mathbb{R}^n), \mathrm{d})$ *is a cochain complex, i.e., that* $\mathrm{d} \circ \mathrm{d} = 0$.