Recall the definition of $\Omega^*$:

- As a vector space over $\mathbb{R}$ it has a basis given by monomials
  
  \[ dx_I := dx_{i_1} \cdots dx_{i_p} \]

  for $I = (i_1, \ldots, i_p)$ with $1 \leq i_1 < \cdots < i_p \leq n$.

- It is an algebra with multiplication $\wedge : \Omega^* \otimes \Omega^* \to \Omega^*$ given on basis vectors by:
  
  \[ dx_{i_1} \cdots dx_{i_p} \wedge dx_{i_{p+1}} \cdots dx_{i_{p+q}} := \begin{cases} 
    0 & \text{if } i_k = i_{p+l} \text{ for some } 1 \leq k, l \leq n \\
    \text{sgn}(\sigma) \cdot dx_{\sigma(i_{i_1})} \cdots dx_{\sigma(i_{i_{q+q}})} & \text{otherwise},
  \end{cases} \]

  where in the second case $\sigma$ is the unique permutation of the set $\{i_1, \ldots, i_{p+q}\}$ such that $\sigma(i_1) < \cdots < \sigma(i_{p+q})$.

**Exercise 1.** Let $(A, \wedge)$ be any algebra over $\mathbb{R}$ with chosen elements $d_1, \ldots, d_n \in A$ satisfying

\[ d_i \wedge d_j = -d_j \wedge d_i \quad \text{for all } i, j \tag{1} \]

Show that there is an unique $\mathbb{R}$-algebra morphism $\Omega^* \to A$ which sends $dx_i$ to $d_i$ for all $i$.

**Remark.** This is what it means to say that $\Omega^*$ is presented as an $\mathbb{R}$-algebra by the generators $dx_i$ and the relations [1].

**Exercise 2.** Please (re)familiarize yourselves with the notion of a smooth manifold. Determine whether the following topological spaces (with the subspace topology induced from $\mathbb{R}^2$) are smooth manifolds. (warning: trick question)

1. The circle

\[ S^1 := \{z \in \mathbb{C} : |z| = 1\} \subset \mathbb{C} \cong \mathbb{R}^2; \]

2. the coordinate axes

\[ K := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 x_2 = 0\}; \]

3. the subsets

\[ U := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 < x_2\}, \]

\[ \overline{U} := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \leq x_2\} \]

of (strictly) above-diagonal points in $\mathbb{R}^2$;

4. the intersections $K \cap S^1$, $K \cap U$ and $K \cap \overline{U}$. 

Recall the definition of the de Rham algebra $\Omega^*(\mathbb{R}^n) := C^\infty(\mathbb{R}^n) \otimes_{\mathbb{R}} \Omega^*$ and of the de Rham differential $\Omega^*(\mathbb{R}^n) \to \Omega^{*+1}(\mathbb{R}^n)$ given on elementary tensors by

$$f_I \otimes dx_I \mapsto \sum_{i=1}^n \frac{\partial f_I}{\partial x_i} \otimes dx_i \wedge dx_I$$

and extended $\mathbb{R}$-linearly (note that we would typically omit the $\otimes$ symbol).

**Exercise 3.** Show that $(\Omega^*(\mathbb{R}^n), d)$ is a cochain complex, i.e., that $d \circ d = 0$. 