summer semester 2020

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Advanced Topics in Algebraic Topology — Exercise Sheet 10

Exercise class: Friday, 3th of July, 11-12

Website with further material, including exercise sheets:
`https://www.groups.ma.tum.de/algebra/scheimbauer/advanced-topics-in-algebraic-topology/`

Fix a manifold B.

Exercise 1. Convince yourself that, for each q, that there is a canonical identification of $\mathrm{C}^\infty(B)$-modules

$$\Omega^q(B) \cong \Gamma(B, \Lambda^q \mathrm{T}^\vee(B)),$$

where $\mathrm{T}(B) \to B$ denotes the tangent bundle, $\mathrm{T}^\vee(B) \to B$ is its $\mathbb{R}$-dual (called the cotangent bundle) and Λ^q denotes the q-th exterior power. In other words: differential forms are the same thing as sections of the exterior power of the cotangent bundle. For a possible reference, see Lee: "Introduction to smooth manifolds", pages 359–362.

Let G be a Lie group. If you don't know what this means, just imagine $G = \mathrm{GL}_\mathrm{k}(\mathbb{R})$ or $G = \mathrm{O}_\mathrm{k}(\mathbb{R})$.

Definition 1 (G-cocycles and equivalence)**.** A G-cocycle defined on an open cover $\{U_\alpha\}_{\alpha \in A}$ of B is a collection of smooth map

$$\tau_{\alpha\beta} \colon U_\alpha \cap U_\beta \longrightarrow G$$

(for all $\alpha, \beta \in A$) such that we have

$$\tau_{\alpha\beta} \cdot \tau_{\beta\gamma} = \tau_{\alpha\gamma}$$

on $U_\alpha \cap U_\beta \cap U_\gamma$ (for all $\alpha, \beta, \gamma \in A$).

Consider two G-cocycles $\{\tau''_{\alpha\beta}\}_{\alpha,\beta}$ and $\{\tau'_{\alpha\beta}\}_{\alpha,\beta}$, defined on open covers $\{U''_\alpha\}_{\alpha \in A''}$ and $\{U'_\alpha\}_{\alpha \in A'}$, respectively. They are called **equivalent**, if there are

- a common refinement of $\{U'_\alpha\}$ and $\{U''_\alpha\}$, i.e. an open cover $\{U_\alpha\}_{\alpha \in A}$ of B and functions $f' \colon A \to A'$ and $f'' \colon A \to A''$ such that, for all $\alpha \in A$, we have $U_\alpha \subseteq U'_{f'(\alpha)} \cap U''_{f''(\alpha)}$;
- a collection of smooth maps

  $$\sigma_\alpha \colon U_\alpha \longrightarrow G$$

  (for each $\alpha \in A$) such that we have

  $$\tau'_{f'(\alpha),f'(\beta)} = \sigma_\alpha \cdot \tau''_{f''(\alpha),f''(\beta)} \cdot \sigma_\beta^{-1}$$

  on $U_\alpha \cap U_\beta$ (for all $\alpha, \beta \in A$).

Exercise 2. Fix a natural number k.

(a) Let $E \to B$ be a real vector bundle of rank k. Let $\{U_\alpha\}_\alpha$ be an open cover of B on which π is trivializable and choose trivializations $\phi_\alpha \colon \pi^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times \mathbb{R}^k$. Show that the transition functions

$$\overline{\tau}_{\alpha\beta} \colon (U_\alpha \cap U_\beta) \times \mathbb{R}^k \xrightarrow{\phi_\beta^{-1}|_{\dots}} \pi^{-1}(U_\alpha \cap U_\beta) \xrightarrow{\phi_\alpha|_{\dots}} (U_\alpha \cap U_\beta) \times \mathbb{R}^k$$

induce a $\mathrm{GL_k}(\mathbb{R})$-cocycle $\tau(E) := \{\tau_{\alpha\beta}\}_{\alpha,\beta}$ satisfying $\overline{\tau}_{\alpha\beta}(p,x) = (p, \tau_{\alpha\beta}(p) \cdot x)$.

(b) Show that, up to equivalence, the cocycle $\tau(E)$ does not depend on the cover $\{U_\alpha\}_\alpha$, nor on the trivializations ϕ_α.

(c) Let ω be a $\mathrm{GL_k}(\mathbb{R})$-cocycle on B. Construct a real vector bundle $E \to B$ of rank k such that $\omega = \tau(E)$. (Hint: If $\omega = \{\omega_{\alpha\beta}\}$ is defined on an open cover $\{U_\alpha\}_{\alpha \in A}$, define E as a suitable quotient of $\coprod_{\alpha \in A}(U_\alpha) \times \mathbb{R}^k$).

(d) Let $E \to B$ and $E' \to B$ be two real vector bundles or rank k. Show that the vector bundles E and E' are isomorphic over B if and only if the associated cocycles $\tau(E)$ and $\tau(E')$ are equivalent.

Conclude that the cocycle construction yields a canonical bijection

$$\{\text{real vector bundles of rank } k \text{ on B}\}/\text{isomorphism} \xleftrightarrow{\cong} \{\mathrm{GL_k}(\mathbb{R})\text{-cocycles on } B\}/\text{equivalence}.$$

Definition 2 (real and complex topological K-theory)**.** Let B be a manifold. We denote by $\mathrm{KO}^0(B)$ the abelian group (additively written) generated by isomorphism classes $[V]$ of real vector bundles on B (of arbitrary finite rank), subject to the relation $[V] + [W] = [V \oplus W]$ (where $\oplus$ denotes the direct sum of vector bundles); it is called the (zeroth) **topological K-theory** of B. We denote by $\widetilde{\mathrm{KO}}^0(B)$ the quotient of $\mathrm{KO}^0(B)$ by the subgroup generated by trivial bundles and call it the **reduced real topological K-theory** of B.
Similarly, we define the **complex K-theory** $\mathrm{KU}^0(B)$ of B by considering *complex* vector bundles on B and the **reduced complex K-theory** $\widetilde{\mathrm{KU}}^0(B)$ by then quotienting out the trivial complex vector bundles.

Exercise 3. (a) Compute all four versions of topological K-theory over a point, i.e. compute $\mathrm{KO}^0(\mathbb{R}^0)$, $\widetilde{\mathrm{KO}}^0(\mathbb{R}^0)$, $\mathrm{KU}^0(\mathbb{R}^0)$ and $\widetilde{\mathrm{KU}}^0(\mathbb{R}^0)$.

(b) Show that every *complex* vector bundle over the circle S^1 is trivial and conclude that $\widetilde{\mathrm{KU}}^0(\mathrm{S}^1)$ is zero. What is $\mathrm{KU}^0(\mathrm{S}^1)$?

Exercise 4 (Bonus challenge)**.** Compute the (reduced and unreduced) real topological K-theory of the circle, i.e. compute $\mathrm{KO}^0(\mathrm{S}^1)$ and $\widetilde{\mathrm{KO}}^0(\mathrm{S}^1)$.
(Hint: Show that for each $k \geq 1$, the group $\mathrm{GL_k}(\mathbb{R})$ has two exactly two path components $\{\det > 0\}$ and $\{\det < 0\}$. Conclude that, up to isomorphism, the real vector bundles of rank k on S^1 are precisely the trivial bundle $\underline{\mathbb{R}^k}$ and the sum $\underline{\mathbb{R}^k} \oplus M$, where $M \to \mathrm{S}^1$ is the Möbius strip)

Remark. As the notation suggests, there are real/complex topological K-groups $\mathrm{KO}^n(B)/\mathrm{KU}^n(B)$ not just in degree $n = 0$, but for each integer n; they assemble to so called cohomology theories assigning invariants to manifolds or even more general topological spaces. Unlike singular cohomology (which you may have seen last semester) or deRham cohomology (which you definitely saw this semester), their value on a point is not trivial anymore, for instance we have $\mathrm{KO}^{-1}(\mathbb{R}^0) = \widetilde{\mathrm{KO}}^{-1}(\mathrm{S}^0) = \widetilde{\mathrm{KO}}^0(\mathrm{S}^1) \neq 0$ (the first two equalities hold for every cohomology theory; you may conclude the final inequality in the bonus challenge).