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Advanced Topics in Algebraic Topology — Exercise Sheet 11

Exercise class: Friday, 10th of July, 11-12

Website with further material, including exercise sheets:
https://www.groups.ma.tum.de/algebra/scheimbauer/advanced-topics-in-algebraic-topology/

In the lecture you saw that for an orientable real vector bundle $E \to M$ of rank k over an orientable manifold M of finite type, there is a canonical isomorphism

$$\mathrm{H}_c^\star(E) \cong \mathrm{H}_c^{\star-k}(M) \tag{1}$$

(note the index shift).

Exercise 1. Find a counterexample to (1) if one drops the orientability assumption.

The following exercise finishes the proof of (1):

Exercise 2. Show that if $E \to M$ is an orientable vector bundle and M is an orientable manifold, then E is an orientable manifold.

Exercise 3. Given a real vector bundle $E \to M$ of rank k, we define the **determinant bundle of** E to be the k-th exterior power $(\det E) := \Lambda^k E \to M$ of E. Prove:

(a) The determinant of E is a line bundle.

(b) The bundle E is orientable if and only if its determinant bundle $(\det E)$ is trivial.

Can you explain why the determinant bundle deserves its name?

Exercise 4. Show that a manifold M is orientable if and only if its tangent bundle $\mathrm{T}M \to M$ is an orientable bundle.

Recall that a principal G bundle is a fiber bundle $P \to M$ with a smooth action $P \times G \to P$ which is free and transitive on each fiber.

Exercise 5. Show that a principal G-bundle $p\colon P \to M$ is trivial if and only if it admits a global section, i.e. a smooth map $s\colon M \to P$ such that $p \circ s = \mathrm{Id}_M$.

Exercise 6. Fix a manifold M and a natural number k. Construct a canonical correspondence between:

- vector bundles of rank k on M (up to iso) and
- principal $\mathrm{GL_k}(\mathbb{R})$ bundles on M (up to iso).

which sends a vector bundle to its associated frame bundle. For the inverse operation, take a principal $\mathrm{GL_k}(\mathbb{R})$ bundle $P \to M$ and use it to construct a vector bundle with total space

$$P \times_{\mathrm{GL_k}(\mathbb{R})} \mathbb{R}^k :== \frac{P \times \mathbb{R}^k}{\forall g \in \mathrm{GL_k}(\mathbb{R}) : (p, v) \sim (p.g, gv)}$$

(here $(p, g) \mapsto p.g$ denotes the right action of $\mathrm{GL_k}(\mathbb{R})$ on P).