Advanced Topics in Algebraic Topology — Exercise Sheet 11

Exercise class: Friday, 10th of July, 11-12

Website with further material, including exercise sheets:
https://www.groups.ma.tum.de/algebra/scheimbauer/advanced-topics-in-algebraic-topology/

In the lecture you saw that for an orientable real vector bundle $E \to M$ of rank $k$ over an orientable manifold $M$ of finite type, there is a canonical isomorphism
\[ H_c^*(E) \cong H_c^{*-k}(M) \] (1)
(note the index shift).

Exercise 1. Find a counterexample to (1) if one drops the orientability assumption.

The following exercise finishes the proof of (1):

Exercise 2. Show that if $E \to M$ is an orientable vector bundle and $M$ is an orientable manifold, then $E$ is an orientable manifold.

Exercise 3. Given a real vector bundle $E \to M$ of rank $k$, we define the determinant bundle of $E$ to be the $k$-th exterior power $(\det E) := \wedge^k E \to M$ of $E$. Prove:
(a) The determinant of $E$ is a line bundle.

(b) The bundle $E$ is orientable if and only if its determinant bundle $(\det E)$ is trivial.

Can you explain why the determinant bundle deserves its name?

Exercise 4. Show that a manifold $M$ is orientable if and only if its tangent bundle $TM \to M$ is an orientable bundle.

Recall that a principal $G$ bundle is a fiber bundle $P \to M$ with a smooth action $P \times G \to P$ which is free and transitive on each fiber.

Exercise 5. Show that a principal $G$-bundle $p: P \to M$ is trivial if and only if it admits a global section, i.e. a smooth map $s: M \to P$ such that $p \circ s = \text{Id}_M$.

Exercise 6. Fix a manifold $M$ and a natural number $k$. Construct a canonical correspondence between:
• vector bundles of rank $k$ on $M$ (up to iso) and
• principal $\text{GL}_k(\mathbb{R})$ bundles on $M$ (up to iso).

which sends a vector bundle to its associated frame bundle. For the inverse operation, take a principal $\text{GL}_k(\mathbb{R})$ bundle $P \to M$ and use it to construct a vector bundle with total space
\[ P \times \text{GL}_k(\mathbb{R}) \mathbb{R}^k := \frac{P \times \mathbb{R}^k}{\forall g \in \text{GL}_k(\mathbb{R}): (p, v) \sim (p.g, gv)} \]
(here $(p, g) \mapsto p.g$ denotes the right action of $\text{GL}_k(\mathbb{R})$ on $P$).