summer semester 2020

Claudia Scheimbauer
Tashi Walde

Advanced Topics in Algebraic Topology — Exercise Sheet 12

Exercise class: Friday, 17th of July, 11-12

Website with further material, including exercise sheets:
https://www.groups.ma.tum.de/algebra/scheimbauer/advanced-topics-in-algebraic-topology/

Let M be an n-dimensional compact oriented manifold. Let $B = \{\beta_1, \ldots, \beta_l\}$ be a homogeneous[1] $\mathbb{R}$-basis of the cohomology $\mathrm{H}^*(M)$. Denote by $\beta^\vee = \{\beta_1^\vee, \ldots, \beta_l^\vee\}$ the dual basis under the Poincaré duality pairing $\int_M - \wedge -\colon \mathrm{H}^*(M) \otimes \mathrm{H}^{n-*}(M) \to \mathbb{R}$; in other words, we have $\int_M \beta_i \wedge \beta_j^\vee = \delta_{ij}$ for all i, j. Note that $\deg(\beta_i^\vee) = n - \deg(\beta_i)$ for all i.

Consider the manifold $M \times M$ with its two projections $p_1, p_2 \colon M \times M \to M$. Let $\Delta \subset M \times M$ be the diagonal, i.e. the submanifold given by the points of the form $(x, x) \in M \times M$; note that $M \cong \Delta$ via $x \mapsto (x, x)$

Exercise 1. Show that the Poincaré dual $\eta_\Delta \in \mathrm{H}^n(M \times M)$ of the submanifold $\Delta \subset M \times M$ is given by the formula

$$\eta_\Delta = \sum_{j=1}^{l} (-1)^{\deg(\beta_i)} p_1^*(\beta_j) \wedge p_2^*(\beta_j^\vee).$$

Conclude that we have $\int_\Delta \eta_\Delta = \chi(M)$ (the Euler characteristic of M).

Given a manifold M and a submanifold $S \subset M$, we define a certain vector bundle $\mathrm{N}(S/M)$ on S, called the **normal bundle of S in M** as the cokernel of the inclusion $\mathrm{T}S \hookrightarrow \mathrm{T}M|_S$. In other words, the fiber of $\mathrm{N}(S/M)$ at $p \in S$ is the quotient of vector spaces $\mathrm{N}_p(S/M) := \frac{\mathrm{T}_p M}{\mathrm{T}_p S}$.

Exercise 2. Show that the normal bundle $\mathrm{N}(\Delta/(M \times M))$ of $\Delta \subset M \times M$ is isomorphic to $\mathrm{T}M$ as vector bundles on $M \cong \Delta$.

A fiber bundle whose fiber is the k-sphere S^k is called a k-dimensional **sphere bundle**.

Exercise 3. Let $E \to B$ be a real vector bundle of rank k. Construct a k-dimensional sphere bundle $\hat{E} \to B$ by adding a "point at infinity" to each fiber E_p (for $p \in B$). More precisely the sphere bundle $\hat{E} \to B$ is supposed to have a global section $\infty : B \to \hat{E}$ and an identification $\hat{E} \setminus \infty(B) \cong E$ of bundles over B.
What manifold $\hat{E}$ does this construction produce in the case where $E \to \mathrm{S}^1$ is the Möbius strip?

---

[1] Homogeneous means $\beta_j \in \mathrm{H}^r(M)$ for some $r = \deg(\beta_j)$.