Let $M$ be an $n$-dimensional compact oriented manifold. Let $B = \{\beta_1, \ldots, \beta_l\}$ be a homogeneous $\mathbb{R}$-basis of the cohomology $H^*(M)$. Denote by $\beta^\vee = \{\beta^\vee_1, \ldots, \beta^\vee_l\}$ the dual basis under the Poincaré duality pairing $\int_M - \wedge - : H^r(M) \otimes H^{n-r}(M) \to \mathbb{R}$; in other words, we have $\int_M \beta_i \wedge \beta^\vee_j = \delta_{ij}$ for all $i, j$. Note that $\deg(\beta^\vee_j) = n - \deg(\beta_i)$ for all $i$.

Consider the manifold $M \times M$ with its two projections $p_1, p_2 : M \times M \to M$. Let $\Delta \subset M \times M$ be the diagonal, i.e. the submanifold given by the points of the form $(x, x) \in M \times M$; note that $M \cong \Delta$ via $x \mapsto (x, x)$

**Exercise 1.** Show that the Poincaré dual $\eta_\Delta \in H^n(M \times M)$ of the submanifold $\Delta \subset M \times M$ is given by the formula

$$\eta_\Delta = \sum_{j=1}^l (-1)^{\deg(\beta_i)} p_1^*(\beta_j) \wedge p_2^*(\beta^\vee_j).$$

Conclude that we have $\int_\Delta \eta_\Delta = \chi(M)$ (the Euler characteristic of $M$).

Given a manifold $M$ and a submanifold $S \subset M$, we define a certain vector bundle $N(S/M)$ on $S$, called the **normal bundle of $S$ in $M$** as the cokernel of the inclusion $TS \hookrightarrow TM|_S$. In other words, the fiber of $N(S/M)$ at $p \in S$ is the quotient of vector spaces $N_p(S/M) := \frac{T_pM}{T_pS}$.

**Exercise 2.** Show that the normal bundle $N(\Delta/(M \times M))$ of $\Delta \subset M \times M$ is isomorphic to $TM$ as vector bundles on $M \cong \Delta$.

A fiber bundle whose fiber is the $k$-sphere $S^k$ is called a **$k$-dimensional sphere bundle**.

**Exercise 3.** Let $E \to B$ be a real vector bundle of rank $k$. Construct a $k$-dimensional sphere bundle $\tilde{E} \to B$ by adding a “point at infinity” to each fiber $E_p$ (for $p \in B$). More precisely the sphere bundle $\tilde{E} \to B$ is supposed to have a global section $\infty : B \to \tilde{E}$ and an identification $\tilde{E} \setminus \infty(B) \cong E$ of bundles over $B$.

What manifold $\tilde{E}$ does this construction produce in the case where $E \to S^1$ is the Möbius strip?

\[\text{Homogeneous means } \beta_j \in H^r(M) \text{ for some } r = \deg(\beta_j).\]