summer semester 2020

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Advanced Topics in Algebraic Topology — Exercise Sheet 2

Exercise class: Friday, 8th May; 10-12

Website with further material, including exercise sheets:
https://www.groups.ma.tum.de/algebra/scheimbauer/advanced-topics-in-algebraic-topology/

Recall that a map $f\colon X \to \mathbb{R}$ (on a topological space) is called **locally constant** if each point $x \in X$ has a neighborhood on which f is constant.

Exercise 1. *Let $U \subset \mathbb{R}^n$ be an open subset.*

(1) Prove that a continous map $f\colon U \to \mathbb{R}$ is locally constant if and only if it is constant on each connected component of U.

(2) Given two smooth functions $f, g\colon U \to \mathbb{R}$, show that $\mathrm{d}f = \mathrm{d}g$ if and only if $f - g$ is locally constant.

Recall that a 1-form ω is called **closed** if $\mathrm{d}\omega = 0$ and **exact** if there exists a function f such that $\mathrm{d}f = \omega$.

Exercise 2. *Consider the circle $\mathrm{S}^1 := \{z : |z| = 1\} \subset \mathbb{C}$ with the standard charts*

$$\phi_+^{-1}\colon \qquad (0, 2\pi) \xrightarrow{\cong} \mathrm{S}^1 \setminus \{1\}$$

$$\phi_-^{-1}\colon \qquad (-\pi, \pi) \xrightarrow{\cong} \mathrm{S}^1 \setminus \{-1\}$$

given by $x \mapsto e^{ix}$. Show that $\phi_+^*(\mathrm{d}x) \in \Omega^1(\mathrm{S}^1 \setminus \{1\})$ and $\phi_-^*(\mathrm{d}x) \in \Omega^1(\mathrm{S}^1 \setminus \{-1\})$ glue to a well defined 1-form ω on S^1. Is ω closed? Is ω exact?

Exercise 3. *Consider the following 1-form on $\mathbb{R}^2 \setminus \{0\}$:*

$$\omega := \frac{-y\mathrm{d}x + x\mathrm{d}y}{x^2 + y^2} \tag{1}$$

(1) Show that ω is closed on $\mathbb{R}^2 \setminus \{0\}$.

(2) On which of the following open subsets $\mathbb{R}^2 \setminus \{0\}$ is ω exact?

(a) All of $\mathbb{R}^2 \setminus \{0\}$.

(b) The open annulus $\{(x, y) : 1 < x^2 + y^2 < 2\}$.

(c) The upper half plane $\{(x, y) : y > 0\}$.

(d) The right half plane $\{(x, y) : x > 0\}$.

(e) The lower half plane $\{(x, y) : y < 0\}$.

(f) The complement of the negative x-axis: $\mathbb{R}^2 \setminus \{(x, 0) : x < 0\}$.

Hint: Consider the function $\theta\colon \mathbb{C} \setminus \{0\} \to \mathbb{R}/2\pi\mathbb{Z}$ given by $re^{i\theta} \mapsto \theta$.

Exercise 4. *Determine whether the* 1*-form*

$$\frac{x\mathrm{d}x + y\mathrm{d}y}{(x^2 + y^2)^2}$$

on $\mathbb{R}^2 \setminus \{0\}$ *is exact.*

Exercise 5. *Let* $U \subset \mathbb{R}^2$ *be an open disk of some radius* r *around some center* (x, y)*. Prove that every closed* 1*-form on* U *is exact.*