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Advanced Topics in Algebraic Topology — Exercise Sheet 3

Exercise class: Friday, 15th of May, 11-12

Website with further material, including exercise sheets:
`https://www.groups.ma.tum.de/algebra/scheimbauer/advanced-topics-in-algebraic-topology/`

Exercise 1 (snake lemma)**.** *Consider the commutative solid diagram*

$$\begin{array}{ccccccccc} 0 & \dashrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \dashrightarrow & 0 \end{array} \tag{1}$$

(of modules over some ring) and assume that both rows are exact. Show that it induces a "long" exact sequence:

$$0 \dashrightarrow \ker a \xrightarrow{f} \ker b \xrightarrow{g} \ker c \longrightarrow \operatorname{coker} a \xrightarrow{f'} \operatorname{coker} b \xrightarrow{g'} \operatorname{coker} c \dashrightarrow 0 \tag{2}$$

If the rows in the original diagram (1) *are exact also with the dashed arrows, then the same is true for the long exact sequence* (2).

Exercise 2. *(long exact sequence in (co)homology) Let*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{3}$$

be a short exact sequence of cochain complexes (of modules over some ring). Use the snake lemma to construct the long exact sequence of cohomology which looks as follows:

$$\cdots \longrightarrow \mathrm{H}^*(A) \xrightarrow{f^\star} \mathrm{H}^*(B) \xrightarrow{g^\star} \mathrm{H}^*(C) \xrightarrow{\partial} \mathrm{H}^{*+1}(A) \xrightarrow{f^\star} \mathrm{H}^{*+1}(B) \xrightarrow{g^\star} \mathrm{H}^{*+1}(C) \longrightarrow \cdots \tag{4}$$

Exercise 3. *(Mayer-Vietoris sequence) Let U and V be open subsets of some manifold. Use Exercise 2 to construct the so called Mayer-Vietoris long exact sequence in de Rham cohomology:*

$$\cdots \longrightarrow \mathrm{H}^*_{\mathrm{dR}}(U \cup V) \longrightarrow \mathrm{H}^*_{\mathrm{dR}}(U) \oplus \mathrm{H}^*_{\mathrm{dR}}(V) \longrightarrow \mathrm{H}^*_{\mathrm{dR}}(U \cap V) \xrightarrow{\partial} \mathrm{H}^{*+1}_{\mathrm{dR}}(U \cup V) \longrightarrow \mathrm{H}^{*+1}_{\mathrm{dR}}(U) \oplus \mathrm{H}^{*+1}_{\mathrm{dR}}(V) \longrightarrow \mathrm{H}^{*+1}_{\mathrm{dR}}(U \cap V) \longrightarrow \cdots \tag{5}$$

Give an explicit description of the connecting map $\partial \colon \mathrm{H}^*_{\mathrm{dR}}(U \cap V) \to \mathrm{H}^{*+1}_{\mathrm{dR}}(U \cup V)$.

Exercise 4. *Use the Mayer-Vietoris sequence to compute the de Rham cohomology $\mathrm{H}^*_{\mathrm{dR}}(\mathrm{S}^1)$ of the circle. Provide an explicit basis!*

Exercise 5. *Follow in the footsteps of the previous exercises to construct a Mayer-Vietoris sequence for de Rham cohomology* with compact support*. Use it to explicitly compute the de Rham cohomology with compact support of* S^1*.*