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Advanced Topics in Algebraic Topology — Exercise Sheet 4

Exercise class: Friday, 22th of May, 11-12

Exercise 1. *Let ω be an $(n-1)$-form on $\mathbb{R}^n$ with compact support.*

(1) Show that $\int_{\mathbb{R}^n} \mathrm{d}\omega = 0$.

(2) Is the same true if we replace $\mathbb{R}^n$ by the upper half space $\{(x_1, \ldots, x_n) : x_n \geq 0\}$?

(3) What happens if we drop the assumption that ω is compactly supported?

Exercise 2. *Show that S^1 is orientable and that, for each choice of orientation, the integration map $\int_{\mathrm{S}^1} \colon \Omega^1(\mathrm{S}^1) = \Omega_c^1(\mathrm{S}^1) \to \mathbb{R}$ descends to an isomorphism $\mathrm{H}^1(\mathrm{S}^1) = \mathrm{H}_c^1(\mathrm{S}^1) \xrightarrow{\cong} \mathbb{R}$.*

Exercise 3. *Compute the de Rham cohomology H^* and the de Rham cohomology H_c^* with compact support of*

(1) the open cylinder $\mathrm{S}^1 \times (0, 1)$;

(2) the open Möbius strip $\frac{[0,1] \times (0,1)}{(0,x) \sim (1,1-x)}$;

(3) the torus $\mathrm{S}^1 \times \mathrm{S}^1$.

Exercise 4. *Determine which of the manifolds in Exercise 3 are orientable.*

Exercise 5. *Given a 1-form ω on an open subset $U \subset \mathbb{R}^n$, show that the following are equivalent:*

(1) ω is exact on U;

(2) $\int_\gamma \omega = \int_\delta \omega$ whenever γ and δ are smooth paths in U with the same initial and final points;

(3) $\int_\gamma \omega = 0$ whenever γ is a smooth loop in U.

Exercise 6. *Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be two coordinate systems on $\mathbb{R}^n$. Show: $\mathrm{d}y_1 \ldots \mathrm{d}y_n = \det(\frac{\partial y_i}{\partial x_j})_{i,j} \mathrm{d}x_1 \ldots \mathrm{d}x_n$.*