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Advanced Topics in Algebraic Topology — Exercise Sheet 5

Exercise class: Friday, 29th of May, 11-12

Website with further material, including exercise sheets:
`https://www.groups.ma.tum.de/algebra/scheimbauer/advanced-topics-in-algebraic-topology/`

Recall Stokes' Theorem, which states that for an $(n-1)$-form ω with compact support on an oriented manifold M we have

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega,$$

where ∂M denotes the boundary of M with its induced orientation.

Exercise 1. Prove Stokes' Theorem for the upper n-dimensional half space

$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n \,:\, x_n \geq 0\}$$

with boundary $\mathbb{R}^{n-1}$.

Exercise 2 (Orientation covering)**.** Let M be a connected manifold. The goal of this exercise is to construct a canonical 2-sheeted covering of $\widetilde{M} \to M$ which "encodes" the possible local orientations of M.
Proceed in the following steps (you don't need to check every last detail; just convince yourself that the construction works):

(1) Define $\widetilde{M}$ as the set of pairs (x, O_x), where $x \in M$ and where O_x is a choice of local orientation at x (i.e. an equivalence class of nowhere zero top forms defined on a sufficiently small neighborhood of x).

(2) Equip $\widetilde{M}$ with a topology and a smooth atlas such that the canonical projection $\pi\colon \widetilde{M} \to M$ given by $(x, O_x) \mapsto x$ is a smooth double covering (i.e. each point $x \in M$ has a neighborhood U restricted to which the projection is diffeomorphic to $U \coprod U \to U$).

(3) Equip $\widetilde{M}$ with the tautological (global) orientation O which locally at a point (x, O_x) is given by $\pi^*(O_x)$.

(4) Prove: if M is orientable then $\widetilde{M} \to M$ is diffeomorphic to the trivial double covering $M \coprod M \to M$ and the two components of $\widetilde{M}$ carry opposite orientations.

(5) Prove: if M is not orientable then $\widetilde{M}$ is connected.

(6) If you know covering theory (e.g. from last semester): what is the map $\pi_1(M) \to \mathbb{Z}/2\mathbb{Z}$ associated to the double covering $\pi\colon \widetilde{M} \to M$ in the case where M is not orientable?

Exercise 3 (Volume form on the sphere)**.** Let $n \geq 1$. Let $r\colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}$ be the radius function $r = \sqrt{x_1^2 + \ldots x_n^2}$. For each α, consider the n-form

$$\omega_\alpha \coloneqq \frac{1}{r^\alpha} \sum_{i=0}^{n} (-1)^i x_i \mathrm{d}x_1 \cdots \widehat{\mathrm{d}x_i} \cdots \mathrm{d}x_n$$

(where the hat denotes that $\mathrm{d}x_i$ is missing).

(1) For which α is ω_α is closed?

(2) Compute the integral $\int_{\mathrm{S}^n} \omega_\alpha$ of ω_α over the n-sphere $\mathrm{S}^n := \{r = 1\} \subset \mathbb{R}^{n+1}$.

(3) For which α is ω_α exact?

(4) Compute $\mathrm{d}r \wedge \omega_\alpha$ in $\Omega^*(\mathbb{R} \setminus \{0\})$.

Definition 1. Fix two cochain complexes $(A^\bullet, d_A)$ and $(B^\bullet, d_B)$ and let $f, g \colon A^\bullet \to B^\bullet$ be two chain maps. A **(chain) homotopy** from f to g is a collection of maps $h \colon A^\bullet \to B^{\bullet-1}$ such that $g - f = (d_B \circ h) + (h \circ d_A)$; we write $h \colon f \simeq g$.
A map $f \colon A^\bullet \to B^\bullet$ is called a **(chain) homotopy equivalence** if it "admits an inverse up to homotopy", i.e. if there is a map $f' \colon B^\bullet \to A^\bullet$ and homotopies $f \circ f' \simeq 1_B$ and $f' \circ f \simeq 1_A$.

Exercise 4. Fix two cochain complexes A and B.

(1) Prove that "being chain homotopic" is an equivalence relation on the set of chain maps from A to B.

(2) Show that if two chain maps $f, g \colon A \to B$ are homotopic then they induce induce the same morphism on homology, i.e. $H^*(f) = H^*(g) \colon H^*(A) \to H^*(B)$. Is the converse true?

(3) Show that if a map $f \colon A \to B$ is a chain homotopy equivalence then it induces an isomorphism in homology. Is the converse true?