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Advanced Topics in Algebraic Topology — Exercise Sheet 6

Exercise class: Friday, 5th of June, 11-12

Website with further material, including exercise sheets:
`https://www.groups.ma.tum.de/algebra/scheimbauer/advanced-topics-in-algebraic-topology/`

In the next lecture you will see Poincaré duality, which gives, for each oriented n-dimensional manifold M, a canonical isomorphism

$$\mathrm{H}^*(M) \cong \mathrm{H}_c^{n-*}(M)^\vee,$$

relating the ordinary cohomology to the (dual of the) compactly supported one "turned upside down".

Exercise 1. Now that you have seen the precise statement of the Poincaré lemma in the lecture, please revisit Exercise 3 on Sheet 4 in more detail. Verify Poincaré duality in those examples.

Exercise 2. Using the Mayer-Vietoris argument, the Poincaré lemma, and induction, compute the de Rham cohomology of the n-sphere S^n.

The following two classical results are special cases of Stokes's theorem.

Theorem 1. *(Green's theorem) Let γ be a smooth, positively oriented, simple closed curve in $\mathbb{R}^2$ and let D be the compact region bounded by γ. Let f, g be smooth functions on D. Then we have*

$$\oint_\gamma f \mathrm{d}x_1 + g \mathrm{d}x_2 = \iint_D \left(\frac{\partial g}{\partial x_1} - \frac{\partial f}{\partial x_2} \right) \mathrm{d}x_1 \mathrm{d}x_2.$$

Theorem 2. *(Divergence theorem) Let S be an embedded compact surface in $\mathbb{R}^3$ bounding a compact region V; denote by n the unit normal vector field of S. Let F be a smooth vector field on V. Then we have the relation*

$$\iiint_V (\nabla \cdot F) \mathrm{d}V = \oiint_S (F \cdot n) \mathrm{d}S$$

between a volume integral on V and a surface integral on S.

Exercise 3. Show that Green's theorem and the Divergence theorem are indeed special cases of Stokes'. In the second case you may need to look up the definition of the volume and surface integral and unravel the notation. What do these theorems say geometrically?