Exercise 1. Let $M$ be a compact orientable $n$-dimensional manifold. Show directly, without invoking Poincaré duality, that the top cohomology $H^n(M)$ is not zero. (Hint: consider a nowhere vanishing top form and use Stokes to show that it cannot be exact.)

Recall the following definitions from linear algebra.

Definition 1. • A symmetric bilinear form on a $k$-vector space $V$ is a map $b: V \times V \to k$ which is linear in each component (i.e. for each $x \in V$, both $b(x, -)$ and $b(-, x)$ are linear maps $V \to k$) and satisfying $b(x, y) = b(y, x)$.

The form $b$ is called non-degenerate if it satisfies, for each $x \in V$:

$$x = 0 \iff \forall y \in V : b(x, y) = 0$$

• Let $V$ be a finite dimensional real vector space and $b$ a symmetric bilinear form on $V$. Fix a basis $\{x_1, \ldots, x_n\}$ of $V$ and consider the matrix $A = (b(x_i, x_j))_{i,j}$. The signature of $b$ is the triple $\sigma(b) := (\sigma_0, \sigma_-, \sigma_+)$ where $\sigma_0$, $\sigma_-$ and $\sigma_+$ are the numbers eigenvalues of $A$ (counted with multiplicity) which are zero, negative and positive, respectively. (Why does $\sigma(b)$ not depend on the chosen basis?)

Exercise 2. Fix a positive integer $k$. Let $M$ be an oriented compact connected manifold of dimension $4k$. Use Poincaré duality to construct a canonical non-degenerate symmetric bilinear form on the $\mathbb{R}$-vector space $H^{2k}(M)$. The signature of this bilinear form is called the signature of $M$. What goes wrong if the dimension of $M$ is not divisible by 4?

Exercise 3. Compute the signature of the 4-torus $T^4 := S^1 \times S^1 \times S^1 \times S^1$. (Hint: use that a basis for the cohomology of $T^4$ is given by the forms $d\theta_{i_1} \wedge \cdots \wedge d\theta_{i_k}$ (for $1 \leq i_1 < \cdots < i_k \leq 4$), where $d\theta_i := \pi_i^*(d\theta)$ is the pullback of the angular 1-form on the circle along the $i$-th projection $\pi_i: T^4 \to S^1$.

Definition 2. Whenever the cohomology of a manifold $M$ is finite dimensional (e.g. when $M$ is compact), we define

• the Betti numbers of $M$ are the dimensions $\beta^i_M := \dim_{\mathbb{R}} H^i(M)$ for $i = 0, \ldots, \dim M$;
• the Poincaré polynomial of $M$ is $P_M := \sum_{i=0}^{\dim M} \beta^i_M X^i \in \mathbb{Z}[X]$;
• the Euler characteristic of $M$ is the alternating sum $\chi(M) := P_M(-1) = \sum_{i=0}^{\dim M} (-1)^i \beta^i_M$.

Exercise 4. Show that the Euler characteristic of a compact, orientable, odd-dimensional manifold is zero.

\[\text{Sometimes the signature of } b \text{ is defined to be just the integer } \sigma_+ - \sigma_-\]
**Exercise 5** (Bonus challenge). What happens if one drops the orientability assumption in Exercise? 
(Hint: consider the orientation covering from last week’s exercise sheet.)