summer semester 2020

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Advanced Topics in Algebraic Topology — Exercise Sheet 7

Exercise class: Friday, 12th of June, 11-12

Website with further material, including exercise sheets:
`https://www.groups.ma.tum.de/algebra/scheimbauer/advanced-topics-in-algebraic-topology/`

Exercise 1. Let M be a compact orientable n-dimensional manifold. Show directly, without invoking Poincaré duality, that the top cohomology $\mathrm{H}^n(M)$ is not zero. (Hint: consider a nowhere vanishing top form and use Stokes to show that it cannot be exact.)

Recall the following definitions from linear algebra.

Definition 1.

- A **symmetric bilinear form** on a k-vector space V is a map $b\colon V \times V \to k$ which is linear in each component (i.e. for each $x \in V$, both $b(x,-)$ and $b(-,x)$ are linear maps $V \to k$) and satisfying $b(x,y) = b(y,x)$.

  The form b is called **non-degenerate** if it satisfies, for each $x \in V$:

  $$x = 0 \iff \forall y \in V : b(x,y) = 0$$

- Let V be a finite dimensional real vector space and b a symmetric bilinear form on V. Fix a basis $\{x_1, \ldots, x_n\}$ of V and consider the matrix $A = (b(x_i, x_j))_{i,j}$. The **signature**[1] of b is the triple $\sigma(b) \coloneqq (\sigma_0, \sigma_-, \sigma_+)$ where σ_0, σ_- and σ_+ are the numbers eigenvalues of A (counted with multiplicity) which are zero, negative and positive, respectively. (Why does $\sigma(b)$ not depend on the chosen basis?)

Exercise 2. Fix a positive integer k. Let M be an oriented compact connected manifold of dimension $4k$. Use Poincaré duality to construct a canonical non-degenerate symmetric bilinear form on the $\mathbb{R}$-vector space $\mathrm{H}^{2k}(M)$. The signature of this bilinear form is called the **signature of** M . What goes wrong if the dimension of M is not divisible by 4?

Exercise 3. Compute the signature of the 4-torus $T^4 \coloneqq \mathrm{S}^1 \times \mathrm{S}^1 \times \mathrm{S}^1 \times \mathrm{S}^1$. (Hint: use that a basis for the cohomology of T^4 is given by the forms $\mathrm{d}\theta_{i_1} \wedge \cdots \wedge \mathrm{d}\theta_{i_k}$ (for $1 \leq i_1 < \cdots < i_k \leq 4$), where $\mathrm{d}\theta_i \coloneqq \pi_i^\star(\mathrm{d}\theta)$ is the pullback of the angular 1-form on the circle along the i-th projection $\pi_i\colon T^4 \to \mathrm{S}^1$.

Definition 2. Whenever the cohomology of a manifold M is finite dimensional (e.g. when M is compact), we define

- the **Betti numbers** of M are the dimensions $\beta^i_M \coloneqq \dim_{\mathbb{R}} \mathrm{H}^i(M)$ for $i = 0, \ldots, \dim M$;
- the **Poincaré polynomial** of M is $P_M \coloneqq \sum_{i=0}^{\dim M} \beta^i_M X^i \in \mathbb{Z}[X]$;
- the **Euler characteristic** of M is the alternating sum $\chi(M) \coloneqq P_M(-1) = \sum_{i=0}^{\dim M} (-1)^i \beta^i_M$.

Exercise 4. Show that the Euler characteristic of a compact, orientable, odd-dimensional manifold is zero.

[1] Sometimes the signature of b is defined to be just the integer $\sigma_+ - \sigma_-$.

Exercise 5 (Bonus challenge)**.** What happens if one drops the orientability assumption in Exercise 4? (Hint: consider the orientation covering from last week's exercise sheet.)