Exercise 1.  (a) Use the Künneth formula to compute the cohomology of $\mathbb{T}^n := S^1 \times \cdots \times S^1$. Give an explicit basis!

(b) What is the Euler characteristic (as defined on the last sheet) of $\mathbb{T}^n$?

(c) Determine the Poincaré dual $\eta_S \in H^1(\mathbb{T}^2)$ of the submanifold $S := S^1 \times \{1\} \subset \mathbb{T}^2$.

Definition 1 (Real and complex projective lines). The **complex projective line** is the topological space

$$\mathbb{CP}^1 := \frac{\mathbb{C}^2 \setminus \{0\}}{v \sim w \text{ if and only if } v, w \text{ span the same vector subspace } \langle v \rangle_{\mathbb{C}} = \langle w \rangle_{\mathbb{C}} \text{ of } \mathbb{C}^2}.$$ 

Replacing $\mathbb{C}$ by $\mathbb{R}$ yields the **real projective line** $\mathbb{RP}^1$.

Exercise 2.  (a) Equip $\mathbb{RP}^1$ with the structure of a smooth manifold diffeomorphic to $S^1$.

(b) Equip $\mathbb{CP}^1$ with the structure of a smooth manifold which is diffeomorphic to the 2-sphere $S^2$.

Exercise 3 (Hopf fibration).  (a) Show that the projection $p: \mathbb{C}^2 \setminus \{0\} \to \mathbb{CP}^1$ is a non-trivial fiber bundle with fibers diffeomorphic to $\mathbb{C}^\times$.

(b) Show that $p$ contains a non-trivial fiber subbundle $S^3 \to S^2 \cong \mathbb{CP}^1$ with fibers diffeomorphic to $S^1$.

Exercise 4 (Tautological bundle on projective lines).  (1) Construct a non-trivial real vector bundle of rank 1 on $\mathbb{RP}^1$ such that the fiber over each point $[v] \in \mathbb{RP}^1$ is the $\mathbb{R}$-vector space $\langle v \rangle_{\mathbb{R}} \subset \mathbb{R}^2$. Have you seen this bundle before?

(2) Construct a non-trivial complex vector bundle of rank 1 on $\mathbb{CP}^1$ such that the fiber over each point $[v] \in \mathbb{CP}^1$ is the $\mathbb{C}$-vector space $\langle v \rangle_{\mathbb{C}} \subset \mathbb{C}^2$. How does this bundle relate to the one constructed in Exercise 3?

Exercise 5 (Bonus challenge). Let $M$ be a compact non-orientable manifold of dimension $n$. Prove that $H^n(M) = 0$. (Hint: Consider the orientation double cover from sheet 5.)