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Advanced Topics in Algebraic Topology — Exercise Sheet 9

Exercise class: Friday, 26th of June, 11-12

Website with further material, including exercise sheets:
`https://www.groups.ma.tum.de/algebra/scheimbauer/advanced-topics-in-algebraic-topology/`

Let M be an n-dimensional manifold and fix a point $p \in M$. Recall that a derivation at p is an $\mathbb{R}$-linear map $D\colon \mathrm{C}^\infty(M) \to \mathbb{R}$ which satisfies the Leibniz product rule at p, i.e. for all $f, g \in \mathrm{C}^\infty(M)$ we have

$$D(f \cdot g) = D(f) \cdot g(p) + f(p) \cdot D(g).$$

Exercise 1. Let $f, g\colon M \to \mathbb{R}$ be smooth functions and let D be a derivation at $p \in M$. Let U be a chart around p with local coordinates $x_1, \ldots, x_n$.

(a) Prove that $\mathrm{T}_p(M) \subset \mathrm{Hom}_{\mathbb{R}}(\mathrm{C}^\infty(M), \mathbb{R})$ is an $\mathbb{R}$-vector subspace.

(b) If $f(p) = 0 = g(p)$, then $D(f \cdot g) = 0$.

(c) If f is constant, then $D(f) = 0$.

(d) Show that for each $i = 1, \ldots, n$ the linear map $\frac{\partial}{\partial x_i}|_p\colon \mathrm{C}^\infty(M) \to \mathbb{R}$ is a derivation at p.

(e) Show that the linear map $\mathbb{R}^n \to \mathrm{T}_p(M)$ given by $(v_1, \ldots, v_n) \mapsto \sum_{i=1}^n v_i \frac{\partial}{\partial x_i}|_p$ (which depends on the local coordinates x) is injective.

Remark. You have in fact seen in the lecture that the map $\mathbb{R}^n \to \mathrm{T}_p(M)$ of Exercise 1(e) is also sujective, hence it is an isomorphism of $\mathbb{R}$-vector spaces.

Exercise 2. Let $\pi\colon E \to M$ be a vector bundle of rank k. For each open subset $U \subseteq M$, we say that a **section** of E (or of π) over U is a smooth map $s\colon U \to E$ satisfying $\pi \circ s = \mathrm{Id}_U$; we denote by $\Gamma(U, E)$ the set of such sections. Show:

(a) For each open subset $U \subset V \subseteq M$, there is a $\mathrm{C}^\infty(V)$-module structure on the set $\Gamma(U, E)$ given by pointwise scalar multiplication. In particular, $\Gamma(U, E)$ is an $\mathbb{R}$-vector space.

(b) Given open subsets $U' \subseteq U \subseteq V \subseteq M$, the canonical restriction map $\mathrm{res}^U_{U'}\colon \Gamma(U, E) \to \Gamma(U', E)$ is $\mathrm{C}^\infty(V)$-linear.

(c) Let $U \subseteq M$ be an open subset and $\{U_\alpha\}_{\alpha \in A}$ be an open cover of U. Assume that we have, for each $\alpha \in A$, a section $s_\alpha \in \Gamma(U_\alpha, E)$ and that they satisfy $\mathrm{res}^{U_\alpha}_{U_\alpha \cap U_\beta}(s_\alpha) = \mathrm{res}^{U_\beta}_{U_\alpha \cap U_\beta}(s_\beta)$ for each pair of indices α, β. Then there is a unique section $s \in \Gamma(U, E)$ satisfying $\mathrm{res}^U_{U_\alpha}(s) = s_\alpha$ for all $\alpha \in A$.

(d) The $\mathrm{C}^\infty(M)$-rank of $\Gamma(M, E)$, defined as the number

$$\sup\{r \in \mathbb{N} : \exists s_1, \ldots, s_r \in \Gamma(M, E) \text{ linearly independent over } \mathrm{C}^\infty(M)\},$$

is always at most k. It is equal to k if and only if the bundle $E \to M$ is trivial.

Exercise 3 (Hairy ball theorem)**.** The goal of this exercise is to show that for even n, the n-sphere S^n does not admit a nowhere vanishing vector field (= nowhere zero section of the tangent bundle). For $n = 2$ this says that you cannot comb a hairy ball flat.

(1) Consider the linear map $(-1)\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$. Show that the induced map $(-1)\colon \mathrm{S}^n \to \mathrm{S}^n$ (called the antipode) is smoothly homotopic to the identity on S^n if and only if n is odd. (Hint: consider the effect of the antipode map on the top cohomology of S^n).

(2) Assume that X is a nowhere vanishing vector field on S^n. Use X to construct a homotopy between the identity on S^n and the antipode. Conclude that n must be odd. (Hint: For each point p, the tangent vector $X(p)$ tells you in which direction to walk in order to reach the antipodal point $-p$.)

What about the converse: Are there nowhere vanishing vector fields on *odd*-dimensional spheres?