Hurwitzseminar 2021

Claudia Scheimbauer
Eilind Karlsson

Knots

Time and place: September 6 – 10, hopefully in Reitenhaslach

Webseite: https://www.groups.ma.tum.de/algebra/scheimbauer/ ⇒ Lehre ⇒ Knots

1. General

Every talk should be **90 minutes** long. Hopefully, if things go as planned, they will be in person in Reitenhaslach, where we should have at least a whiteboard or blackboard. For a lot of concepts there are nice animations which you can find online (e.g. Numberphile). If you would like to show a short video, you can use a beamer.

Using the outline, first find references which you plan to use. You should always look at several references to see different viewpoints on your topic.

Contact Eilind Karlsson or myself to discuss the outline of your talk. Every talk should contain (roughly):

- 3 examples and a counterexample
- a proof of a mathematical statement
- at least one drawing/illustration/demonstration – with a topic like this, several are usual possible and encouraged! Be creative!

Please prepare a handout for everyone. If you send it to us until at the latest Wednesday, Sep 1, we will print it.

2. Topics

Overview: The references I added are first sources of information. Find your own as well!

(1) Definition of knots and links, knot diagram/projection, Reidemeister moves, first invariants: alternating knots, unknotting number, tricoloring, optional: towards Alexander polynomial [Ada], [Rob], [AN]

(2) Jones polynomial via bracket polynomial [Ada]

(3) Conway's tangles und continued fractions [Ada]

(4) Braids, the braid group, braid knots (every link is closure of a braid), and torus knots [Ada, Chapter 5]

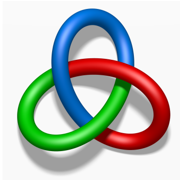
(5) Borromean rings and Brunnian links: "they do not exist in $\mathbb{R}^3$", they are linked [AZ18]

(6) "Applications" to biology, DNA, ... [Ada] [Mur96, Chapter 13], fun database: https://knotprot.cent.uw.edu.pl/

(7) Seifert surface of a knot, knot genus, Seifert matrix, Alexander polynomial and relation to tricoloring (needs classification of surfaces) [Ada], [Mur96, Chapters 5 and 6]

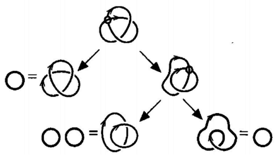
(8) Wirtinger presentation of a knot and computability (needs fundamental group of the knot complement) [Rob, Chapter 8]

(9) Witten's brilliant realization: From Chern-Simons 3 dimensional topological field theory to the Jones polynomial [Wit94] *alternative/additional:* Jones polynomial from 6-vertex model [Mur96, Chapter 12] (both need some basics QFT ideas)



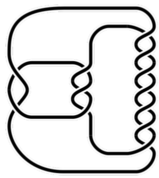
2.1. **Knots, Reidemeister moves, tricoloring.** Explain the definition of knots and links – for this you will have to take an intuitive approach, while being honest about the precise definition (as some students including yourself may not know the mathematically precise definition of embedding and isotopy). Discuss equivalence of knots via ambient isotopy. Explain knot diagrams/projections and Reidemeister moves. If you like, you may sketch why the Reidemeister moves are enough. Introduce some first invariants and types of knots: alternating knots, unknotting number, tricoloring. If time permits, generalize to p-colorings and take first steps towards the Alexander polynomial.

References: [Ada], [Rob], [AN]



2.2. **Jones polynomial and the Kauffman bracket.** Explain the bracket polynomial (Kauffman bracket. Define the Jones polynomial and show that it is a knot invariant. Compute some examples and show that two knots which are not obviously different are indeed different.

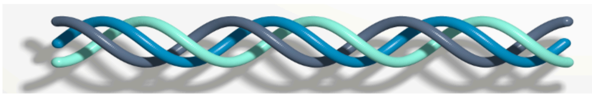
References: [Ada], every standard text on knots!



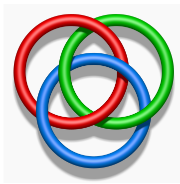
2.3. **Conway's tangles und continued fractions.** Starting with a finite sequence of integers, construct a tangle. Such a tangle is called a *rational tangle*. Explain what a continued fraction is and how to relate rational tangles and continued fractions. You may want to use ropes and helpers to demonstrate the "square dance".

References: [Ada], [GK97]

2.4. **Braids and torus knots.**



Explain what a braid is, define the braid group, and explain the connection. Show that every knot can be obtained as the closure of some braid [Ada]. Introduce torus knots. Explain why $K(p,q) \cong K(q,p)$ [Mur96]. If you have extra time, find some fun facts about braids and/or torus knots.

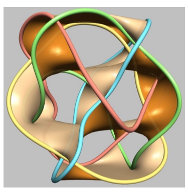


2.5. **Borromean rings and Brunnian links.** Define Brunnian links. Special cases are the Borromean rings and the Tait link. Explain the following non-topological result: Borromean rings do not exist! In this case, we want each ring to lie in a plane. Topologically (if we are allowed to bend the rings) they do exist. Show that they are indeed linked.

References: [AZ18]

2.6. **Applications of knots, e.g. biology.** Explain some "applications" to biology, DNA,... You may have to introduce more concepts from biology rather than show a mathematical proofs. You may find the talk about tangles to be helpful. Explore this topic! There are also plenty of fun videos of knots appearing in nature, but keep in mind that it should still be a mathematical talk.

References: [Ada] [Mur96, Chapter 13], [GK97], [Ric] fun database is here, picture by Javier Arsuaga was taken from this blog post.



2.7. **Seifert surfaces.** Introduce the Seifert surface of a knot, the knot genus, and the Seifert matrix. (Re)define the Alexander polynomial and explain the relation to tricoloring. Do some computations! (You may need to recollect/explain the classification of surfaces.)

References: [Ada], [Mur96, Chapters 5 and 6]

2.8. **Knot groups.**

$$\pi_1(\mathbb{R}^3 \setminus 3_1) = \langle x, y \,|\, xyx = yxy \rangle$$

Introduce the knot group and explain the Wirtinger presentation of a knot. Touch upon computability (You will need to at least heuristically introduce the fundamental group.) [Rob, Chapter 8]

2.9. **Witten's paper: Chern-Simons theory and the Jones polynomial.**

$$S(A) = \frac{k}{4\pi} \int_M \mathrm{Tr} A \wedge dA + \frac{2}{3} A \wedge A \wedge A$$

Explain Witten's brilliant realization that we can recover the Jones polynomial from Chern-Simons 3-dimensional topological field theory [Wit94]

alternative/additional: Jones polynomial from 6-vertex model [Mur96, Chapter 12]

(Both subjects need some basics QFT ideas – be gentle on the audience.)

References

[Ada] C. Adams. The knot book. http://people.math.harvard.edu/~ctm/home/text/books/adams/knot_book/knot_book.pdf.

[AN] Maike Akveld and Otto Neumaier. Die mathematische knotentheorie und ihre aktuellen anwendungen. https://people.math.ethz.ch/~akveld/ArtikelAkveldNeumaier.pdf.

[AZ18] Martin Aigner and Günter M. Ziegler. *Proofs from The Book*. Springer, Berlin, sixth edition, 2018. See corrected reprint of the 1998 original [MR1723092], Including illustrations by Karl H. Hofmann.

[GK97] Jay R. Goldman and Louis H. Kauffman. Rational tangles. *Adv. in Appl. Math.*, 18(3):300–332, 1997.

[Mur96] Kunio Murasugi. *Knot theory and its applications*. Birkhäuser Boston, Inc., Boston, MA, 1996. Translated from the 1993 Japanese original by Bohdan Kurpita.

[Ric] Renzo L. Ricca. *Lectures on Topological Fluid Mechanics*. Access via TUM at https://link-springer-com.eaccess.ub.tum.de/content/pdf/10.1007%2F978-3-642-00837-5.pdf.

[Rob] Justin Roberts. Knots knotes. Available at http://math.ucsd.edu/~justin/Roberts-Knotes-Jan2015.pdf.

[Wit94] Edward Witten. Quantum field theory and the Jones polynomial. In *Braid group, knot theory and statistical mechanics, II*, volume 17 of *Adv. Ser. Math. Phys.*, pages 361–451. World Sci. Publ., River Edge, NJ, 1994.