Sommersemester 2021 Claudia Scheimbauer

Bordisms and Topological Field Theories

Website with further material, including exercise sheets:
`https://www.groups.ma.tum.de/algebra/scheimbauer/` ⇒ Lehre ⇒ Bordisms and Topological Field Theories

1. Topic of course

Studying manifolds up to diffeomorphism is very difficult. However, if we instead study manifolds up to "cobordism" and consider the disjoint union we obtain very computable groups. In fact, product of manifolds gives a cobordism ring. In the 1980's, Atiyah and Segal realized that the notion of cobordism naturally appears when decribing topological field theories mathematically. In the course we will encounter these notions.

2. Outline

Our program will roughly be:

- Introduction to the ideas behind (co)bordism, topological field theories, and how they come up in knot theory via the Jones polynomial
- Cobordism group Ω_n. First examples: 1d oriented and unoriented cobordisms, $\Omega_1^{(or)}$, and Ω_2.
- *Insert:* Smooth manifolds and tangential structures: orientations, possibly string structures
- Two-dimensional case: recollection of classification of 2-dimensional manifolds. Consequences for Ω_3^{or}, outlook on Ω_3 – why algebraic topology quickly becomes necessary
- Cobordism ring and outlook on computations
- Definition of topological field theories (TFTs) after Atiyah and Segal [Ati88]: symmetric monoidal categories and tensor products, the bordism category
- 1d classification and duality
- 2d classification and commutative Frobenius algebras – this will take some time! [Koc04]
- Towards 3-dimensional Chern-Simons/Witten-Reshetikin-Turaev theory – we will follow [KRT97], roughly Chapters 1-5. This needs:
  - Braid groups
  - Representations of braid groups
  - Braided monoidal categories which we will have already introduced
  - Hopf algebras
- *Option:* more advanced cobordism theory stuff. generalized (co)homomology theory, Computation, characteristic classes [Fre]

3. Grade bonus option: outlook talks

If everyone agrees and there are not too many participants, we could try out the following: you could earn a grade bonus by giving a talk about a topic which goes beyond the main content of the course. Topics could include

- Jones polynomial as a knot invariant [Ada]

- Topological quantum field theories from a physics perspective (for a physicist)
- More cobordism theory (for a mathematician with some algebraic topology background)
- Graphical tensor calculus for Frobenius algebras and Hopf algebras
- From a tangle invariant to a TFT: surgery of manifolds.
- generalized (co)homomology theories and (co)bordism
- Variants of the cobordism category: tangles and extra structures
- Towards classifications in higher dimensions (For an advanced student)

⋮

References

[Ada] C. Adams. The knot book. `http://people.math.harvard.edu/~ctm/home/text/books/adams/knot_book/knot_book.pdf`.

[Ati88] Michael Atiyah. Topological quantum field theories. *Inst. Hautes Études Sci. Publ. Math.*, (68):175–186 (1989), 1988.

[Fre] Daniel S. Freed. Bordism: Old and new. `https://web.ma.utexas.edu/users/dafr/bordism.pdf`.

[Koc04] Joachim Kock. *Frobenius algebras and 2D topological quantum field theories*, volume 59 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2004.

[KRT97] Christian Kassel, Marc Rosso, and Vladimir Turaev. *Quantum groups and knot invariants*, volume 5 of *Panoramas et Synthèses [Panoramas and Syntheses]*. Société Mathématique de France, Paris, 1997.