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Algebraic Topology – Exercise 12

(1) (a) Consider the chain complex

$$C_* := \left(\cdots \xleftarrow{2\cdot} \mathbb{Z}/4\mathbb{Z} \xleftarrow{2\cdot} \mathbb{Z}/4\mathbb{Z} \xleftarrow{2\cdot} \cdots \right)$$

Is there a chain homotopy from the identity of C_* to the zero map?

(b) Show that being chain homotopic is an equivalence relation on chain maps.

(2) Simplify in terms of singular homology groups or compute explicitly the following relative homology groups

(a) $H_*^{\text{sing}}(X, \varnothing)$

(b) $H_*^{\text{sing}}(X, X)$

(c) $H_*^{\text{sing}}(X \sqcup Y, Y)$

(3) Consider the embedding $\iota : S^{n-1} \hookrightarrow D^n$. Use the long exact sequence for relative homology to express $H_j^{\text{sing}}(D^n, S^{n-1})$ in terms of homology groups of S^{n-1} for $j > 1$ and $n > 1$.

Definition. Let X be a (non-empty) topological space, and let $\widetilde{C}_*(X)$ be the singular chain complex

$$\widetilde{C}_*(X) := \left(0 \longleftarrow \mathbb{Z} \xleftarrow{\widetilde{\varepsilon}} C_0(X) \longleftarrow C_1(X) \longleftarrow \cdots \right),$$

where $\widetilde{\varepsilon}$ is the map defined in Exercise (4) a) below. We define the *reduced homology* groups to be $\widetilde{H}_*(X) := H_*^{\text{sing}}(\widetilde{C}_*(X))$.

(4) (a) Let X be a topological space. Define a map $\widetilde{\varepsilon} : C_0^{\text{sing}}(X) \to C_0^{\text{sing}}(\{\star\}) \cong \mathbb{Z}$ by $\widetilde{\varepsilon}(\sum_i \lambda_i \alpha_i) = \sum_i \lambda_i$. Show that $\widetilde{\varepsilon}$ makes $\widetilde{C}_*(X)$ into a chain complex and gives a well-defined map on homology.[1]

(b) What is the relation between singular homology and reduced homology for $n \geqslant 1$?

(c) What can you say about $\widetilde{H}_0(X)$?

(d) Using the result from Exercise (6) a), show that we have $\widetilde{H}_n(X) \cong H_n^{\text{sing}}(X, \{x\})$ for any base-point $x \in X$.

[1] Note that this map induces the homomorphism $\varepsilon : H_0(X) \to \mathbb{Z}$ from last weeks lecture.

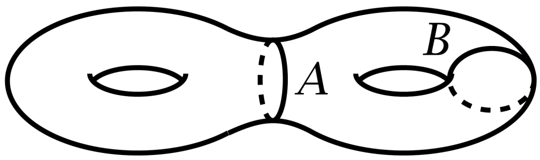
(e) For $\emptyset \neq A \subset X$ we define $\widetilde{H}_n(X, A) := H_n^{\text{sing}}(X, A)$. Construct a long exact sequence of reduced homology, i.e. for any pair of topological spaces (X, A) there is a long exact sequence

$$\cdots \longleftarrow \widetilde{H}_{n-1}(A) \longleftarrow \widetilde{H}_n(X, A) \longleftarrow \widetilde{H}_n(X) \longleftarrow \widetilde{H}_n(A) \longleftarrow \cdots$$

(5) Use the long exact sequence from Exercise (4) (e) to compute. You may use that simplicial and singular homology are isomorphic.

(a) Compute the homology groups $H_n^{\text{sing}}(X, A)$ when X is S^2 or $S^1 \times S^1$ and A is a finite set of points in X.

(b) Compute the homology groups $H_n^{\text{sing}}(X, A)$ and $H_n^{\text{sing}}(X, B)$ for X a closed orientable surface of genus two with A and B the circles indicated in the below Figure.



(6) (a) Using the long exact sequence of relative homology prove that if $\iota : A \hookrightarrow X$ is a retract, then

$$H_n^{\text{sing}}(X) \cong H_n^{\text{sing}}(A) \oplus H_n^{\text{sing}}(X, A), \quad 0 \leqslant n.$$

Hint: For a short exact sequence $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ of abelian groups the following are equivalent:

(i) There exists a homomorphism $s : C \to B$ such that $js = 1_C$, and

(ii) there exists an isomorphism $B \cong A \oplus C$.

(b) Prove that if the inclusion $\iota : A \hookrightarrow X$ is a deformation retract, then for $0 \leqslant n$,

$$H_n^{\text{sing}}(\iota) : H_n^{\text{sing}}(A) \cong H_n^{\text{sing}}(X), \quad H_n^{\text{sing}}(X, A) \cong 0.$$

(7) (a) Let $x \in \mathbb{R}^n$ be an arbitrary point. Express $H_*^{\text{sing}}(\mathbb{R}^n \backslash \{x\})$ in terms of homology groups of spheres.

(b) Let $[y] \in \mathbb{R}P^2$ be an arbitrary point. Express the homology groups $H_*^{\text{sing}}(\mathbb{R}P^2 \backslash \{[y]\})$ in terms of homology groups of spheres.

(c) For $n \in \mathbb{N}$, let $\mathbb{C}P^n := (\mathbb{C}^{n+1} \backslash \{0\}) / \sim$, where $z \sim z'$ if there exists a $\lambda \in \mathbb{C} \backslash \{0\}$ such that $z = \lambda z'$. The spaces $\mathbb{C}P^n$ are called the *complex projective spaces.* Consider an arbitrary point $[z] \in \mathbb{C}P^2$. Express $H_*^{\text{sing}}(\mathbb{C}P^2 \backslash \{[y]\})$ in terms of homology groups of spheres.