Claudia Scheimbauer
Eilind Karlsson

Algebraic Topology – Exercise 13 Sketch of solutions

(1) Compute the homology of $\mathbb{R}^3 \backslash A$, where A is the upper hemisphere of the unit sphere S^2 in $\mathbb{R}^3$.

Solution: Note that A deformation retracts to $\{x\}$ for any point $x \in A$, so their complements in $\mathbb{R}^3$ are homotopy equivalent. (Why?) By homotopy invariance of homology, we need to understand the homology groups of $\mathbb{R}^3 \backslash \{x\}$. Removing one point from $\mathbb{R}^3$ is homotopy equivalent to S^2, so we get

$$H_n(\mathbb{R}^3 \backslash A) \cong H_n(S^2) = \begin{cases} \mathbb{Z}, & n = 0, 2, \\ 0, & \text{otherwise.} \end{cases}$$

(2) Compute the reduced homology of the Klein bottle using the given cover.

Solution: Similarly to the Mayer-Vietoris sequence we have seen in class, one can obtain a reduced version of the Mayer-Vietoris sequence. In more detail, the long exact sequence follows from a short exact sequence of the certain chain complexes (c.f. the proof), so we only need to check that the analogous chain complexes for reduced homology, given by extending the chain complexes by a copy of $\mathbb{Z}$ in degree -1 (c.f. Exercise 4 on Sheet 12), also form a short exact sequence. Do this! It follows that we have a long exact sequence of the form

$$0 \leftarrow \widetilde{H}_0(K) \leftarrow \widetilde{H}_0(X_1) \oplus \widetilde{H}_0(X_2) \leftarrow \widetilde{H}_0(X_1 \cap X_2) \leftarrow \widetilde{H}_1(X) \leftarrow \cdots \tag{1}$$

In this exercise both X_1 and X_2 are Möbius bands, and the same is true for $X_3 := X_1 \cap X_2$. We have seen in class that a Möbius band deformation retracts to S^1 (recall the argument), so for $i \in \{1, 2, 3\}$ we have that

$$\widetilde{H}_n(X_i) \cong \widetilde{H}_n(S^1) = \begin{cases} \mathbb{Z}, & n = 1 \\ 0, & \text{otherwise.} \end{cases}$$

First we note that since $\widetilde{H}_0(X_1) \oplus \widetilde{H}_0(X_2) = 0$ by exactness of (1) we have that $\widetilde{H}_0(K) = 0$. The same argument also ensures that $\widetilde{H}_n(K) = 0$ when $n > 2$. Substituting the known homology groups of X_i into the long exact sequence for $n = 1, 2$ we have the exact sequence

$$0 \leftarrow \widetilde{H}_1(K) \xleftarrow{g} \widetilde{H}_1(X_1) \oplus \widetilde{H}_1(X_2) \cong \mathbb{Z} \oplus \mathbb{Z} \xleftarrow{\widetilde{H}_1(\iota_1, \iota_2)} \widetilde{H}_1(X_1 \cap X_2) \cong \mathbb{Z} \leftarrow \widetilde{H}_2(K) \leftarrow 0 \oplus 0. \tag{2}$$

We need to understand the maps in the long exact sequence to find the unknown homology groups. We have inclusions $\iota_i : X_1 \cap X_2 \hookrightarrow X_i$. Consider a loop in $X_1 \cap X_2$ which is a generator of its first homology group. Under the inclusion into X_i this is sent to a loop running twice along a generator of the first homology of X_i, so on the level of homology groups we have $\widetilde{H}_1(\iota_1, \iota_2)(1) = (2,2)$[1]. Hence, the kernel of this map is 0, so the image of the map $\widetilde{H}_2(K) \to \mathbb{Z}$ in the long exact sequence (2) is the zero map, but since it is also injective by exactness, we find that

$$H_2(K) \cong \widetilde{H}_2(K) = 0.$$

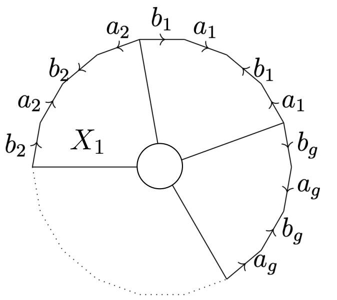
Exactness of (2) also tells us that $\ker(g) = \operatorname{im}(\widetilde{H}_1(\iota_1, \iota_2)) = \mathbb{Z}(2,2)$. Note that $\mathbb{Z}(2,2) \neq 2\mathbb{Z} \oplus 2\mathbb{Z}$! In addition $\operatorname{im}(g) = \ker(0) = \widetilde{H}_1(K)$. Hence we find that

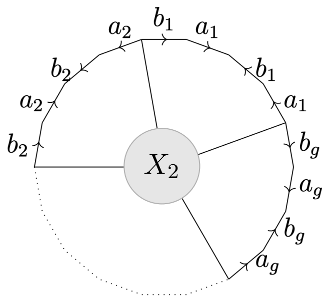
$$\widetilde{H}_1(K) \cong \big(\mathbb{Z}(1,0) \oplus \mathbb{Z}(0,1)\big)/\mathbb{Z}(2,2) \cong \big(\mathbb{Z}(0,1) \oplus \mathbb{Z}(1,1)\big)/2\mathbb{Z}(1,1) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

Remark. The fact that $H_2(K) \cong \widetilde{H}_2(K) = 0$ implies that K is a "non-oriented surface". We briefly talked about orientability in connection with triangulations of surfaces. You may like to think about (or look up) the connection if you are interested.

(3) Use the Mayer-Vietoris Theorem to compute $H_*(F_g)$.

Solution: Choose X_1 and X_2 as in the Figures below, i.e. $X_1 = F_g \backslash \{\text{small disk in the center of } F_g\}$ and $X_2 =$ slightly bigger disk, chosen such that it covers the disk that is removed in X_1.





First we saw in class (recall the argument for this!) that X_1 deformation retracts onto the boundary of F_g which is a wedge sum of circles, i.e. $X_1 \simeq \bigvee_{i=1}^{2g} S^1$. Secondly, we have that $X_2 \cong D^2 \simeq \{*\}$ and $X_1 \cap X_2$ is the complement of a disk in another disk, which deformation retracts to S^1 (Why?). Thus, since homology is homotopy invariant,

$$H_n(X_1) \cong \begin{cases} \mathbb{Z}, & n = 0, \\ \mathbb{Z}^{2g}, & n = 1, \\ 0, & \text{else}; \end{cases} \qquad H_n(X_2) \cong \begin{cases} \mathbb{Z}, & n = 0, \\ 0, & \text{else}; \end{cases} \qquad H_n(X_1 \cap X_2) \cong \begin{cases} \mathbb{Z}, & n = 0, 1, \\ 0, & \text{else}. \end{cases}$$

We use the Mayer-Vietoris sequence for the cover of F_g given by X_1 and X_2.

As in the previous exercise, for $n > 2$, the exact sequence reads

$$0 \cong H_{n-1}(X_1 \cap X_2) \leftarrow H_n(F_g) \leftarrow H_n(X_1) \oplus H_n(X_2) \cong 0 \oplus 0,$$

so by exactness, for $n > 2$, we have $H_n(F_g) \cong 0$.

[1] Depending on how you choose $X_i \cong S^1$ you can also run along X_i in the opposite direction and hence map 1 to e.g. (2,-2) but this does not affect the computation, as long as you are consistent with your choices.

To compute $H_2(F_g)$, consider the following terms in the Mayer-Vietoris sequence:

$$\cdots \leftarrow H_1(F_g) \leftarrow H_1(X_1) \oplus H_1(X_2) \xleftarrow{H_1(\iota_1)} H_1(X_1 \cap X_2) \leftarrow H_2(F_g) \leftarrow 0 \oplus 0 \leftarrow \cdots$$

First, note that by exactness, $H_2(F_g)$ injects into $H_1(X_1 \cap X_2)$. To compute its image, we compute the kernel of $H_1(\iota_1)$.

Since $H_1(X_2) \cong 0$ the map $H_1(\iota_1)$ reduces to a linear map $\bar{\iota} : H_1(X_1 \cap X_2) \cong \mathbb{Z} \to \mathbb{Z}^{2g} \cong H_1(X_1)$ induced from including a loop close to the edge into X_1. On homology, this gives $\bar{\iota}(1) = [a_g, b_g] \cdots [a_1, b_1] = 0$, where the last equality follows from the fact that H_1 is abelian. Thus, from the long exact sequence we have that $H_2(F_g) \cong H_1(X_1 \cap X_2) \cong \mathbb{Z}$.

Continuing the long exact sequence

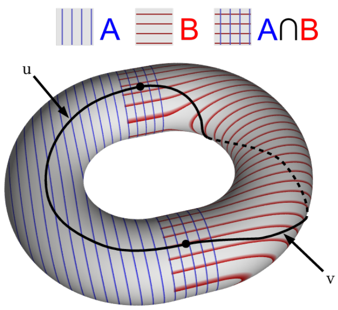
$$\cdots \leftarrow H_0(X_1) \oplus H_0(X_2) \xleftarrow{H_0(\iota_1,\iota_2)} H_0(X_1 \cap X_2) \leftarrow H_1(F_g) \leftarrow H_1(X_1) \oplus H_1(X_2) \xleftarrow{0} \cdots$$

Including $X_1 \cap X_2 \simeq S^1$ into X_1 and X_2 induces the map $H_0(\iota_1, \iota_2)([1]) = ([1], [1])$ on the zeroth homology (Why?). In particular this is an injective map with trivial kernel, so by exactness we deduce that $H_1(F_g) \cong \mathbb{Z}^{2g}$.

Finally, we saw in class that H_0 of a topological space simply counts connected components, of which F_g only has one. Hence, $H_0(F_g) = \mathbb{Z}$. Summarizing, we computed

$$H_*(F_g) = \begin{cases} \mathbb{Z}, & * = 0, 2, \\ \mathbb{Z}^{2g}, & * = 1, \\ 0, & * \geqslant 3. \end{cases}$$

(4) Compute the homology groups of the torus T. *Hint:* Use Mayer-Vietoris, for example with two overlapping cylinders as indicated in the below Figure.



Solution: One can simply plug in $g = 1$ in the results of the previous exercise and we are done. However, practice makes perfect so here we go:

For the given cover we have that A and B are homeomorphic to S^1 times some interval such that the two cylinders slightly overlap. Their intersection $A \cap B$ is homeomorphic to the disjoint union of two copies of S^1 times a small interval. All intervals could be open or closed, but let's chose them to be closed. Hence we already know all of their homology groups since $S^1 \times [a, b]$ deformation retracts onto S^1 (Why?), and we saw in class that the homology of a disjoint union is the direct sum of the homologies. Concretely,

$$H_n(A) = H_n(B) \cong \begin{cases} \mathbb{Z}, & n = 0,1 \\ 0, & \text{else} \end{cases} \qquad H_n(A \cap B) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & n = 0,1 \\ 0, & \text{else.} \end{cases}$$

As in the the previous exercises we find that $H_n(T) = 0$ for $n > 2$ (you could have also used a dimension argument here). Thus, the interesting calculations concern $n = 0, 1, 2$. First we consider the following part of the Mayer-Vietoris sequence

$$\begin{tikzcd}
\cdots \arrow[r, leftarrow] & H_1(T) & H_1(A) \oplus H_1(B) \arrow[l] \arrow[d, "\cong"] & H_1(A \cap B) \arrow[l, "H_1(i,j)"'] \arrow[d, "\cong"] & H_2(T) \arrow[l, "r"'] & H_2(A) \oplus H_2(B) \arrow[l, "0"'] \arrow[d, "\cong"] & \cdots \arrow[l] \\
& & \mathbb{Z} \oplus \mathbb{Z} & \mathbb{Z} \oplus \mathbb{Z} & & 0 \oplus 0 &
\end{tikzcd}$$

We now start by considering the map induced by the inclusions i and j of $A \cap B$ into A and B, respectively. Since $A \cap B$ is a disjoint union of two cylinders it has two generators, one for each cylinder. Denote these two loops by α and β. Similarly, we have one generator each for $H_1(A)$ and $H_1(B)$ (draw this!). We choose the directions of the loops in the following way. (What does this mean in the picture?) We view them in $H_1(A) \oplus H_1(B)$ by concatenating with the inclusion and denote them by the generators $(1, 0)$ and $(0, 1)$ in $\mathbb{Z} \oplus \mathbb{Z}$, respectively. Again using the concatenating with the inclusion into $H_1(A) \oplus H_1(B)$, we have that

$$H_1(i)(\alpha) = (1,0) H_1(j)(\alpha) = (0,1)$$
$$H_1(i)(\beta) = (1,0) H_1(j)(\beta) = (0,1)$$

This depends on the choices of directions of the two loops, and one has to be consistent in these choices. With other choices, signs will appear. The computation on homology does not depend on these choices.

Note that with our choices the two generators α and β are mapped to the same generating circle. Then we have that

$$H_1(i,j)(\alpha) = (1,1) = H_1(i,j)(\beta)$$

and hence $H_1(i,j)(k \cdot \alpha + l \cdot \beta) = (k + l, k + l)$. Hence, we have that $\operatorname{im}(H_1(i,j)) = \mathbb{Z}(1,1) \cong \mathbb{Z}$ and $\ker(H_1(i,j)) = \mathbb{Z}(\alpha - \beta) \cong \mathbb{Z}$. Another way of understanding this map is to note that $H_1(i,j)$ corresponds to the following matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}.$$

This gives us enough information to calculate $H_2(T)$. First, since the part of the long exact sequence above starts with a trivial group we know that r is injective. Exactness gives us $\operatorname{im}(r) = \ker(H_1(i,j)) \cong \mathbb{Z}$, so it follows that $H_2(T) \cong \mathbb{Z}$.

To calculate $H_0(T)$ and $H_1(T)$ we need further pieces of the long exact sequence. To simplify calculations we will work with reduced homology (which only differs from ordinary homology in the zeroth degree). Thus, we consider

$$\begin{tikzcd} 0 & \widetilde{H}_0(T) \arrow[l] & \widetilde{H}_0(A)\oplus\widetilde{H}_0(B) \arrow[l] \arrow[d, "\simeq"] & \widetilde{H}_0(A\cap B) \arrow[l] \arrow[d, "\simeq"] & H_1(T) \arrow[l, "\partial"'] & H_1(A)\oplus H_1(B) \arrow[l, "H_1(f\text{-}g)"'] \arrow[d, "\simeq"] & \cdots \arrow[l, "H_1(i,j)"'] \\ & & 0\oplus 0 & \mathbb{Z} & & \mathbb{Z}\oplus\mathbb{Z} & \end{tikzcd}$$

It follows by exactness that $\widetilde{H}_0(T) = 0$. One can also find $H_0(T)$ by noting that the torus only has one connected component so $H_0(T) = \mathbb{Z}$.

The inclusion of A and B into T are denoted by f and g respectively, and determine the map from the direct sum $H_1(A) \oplus H_1(B)$ to $H_1(T)$. From exactness we have $\ker(H_1(f-g)) = \operatorname{im}(H_1(i,j)) \cong \mathbb{Z}$. Hence,

$$\ker(\partial) = \operatorname{im}(H_1(f-g)) = \mathbb{Z}\oplus\mathbb{Z}/\ker(H_1(f-g)) \cong \mathbb{Z}.$$

Moreover, we know that ∂ is surjective (Why?). So the short exact sequence of the map ∂ becomes

$$\begin{tikzcd} 0 \arrow[r] & \ker(\partial) \arrow[r] \arrow[d, "\simeq"] & H_1(T) \arrow[r] & \operatorname{im}(\partial) \arrow[r] \arrow[d, "\simeq"] & 0 \\ & \mathbb{Z} & & \mathbb{Z} & \end{tikzcd}$$

As $\mathbb{Z}$ is a free abelian group this short exact sequence splits and we can conclude that $H_1(T) \cong \mathbb{Z}\oplus\mathbb{Z}$.[2]

Aside: If you want to solve this without resorting to reduced homology you need to work out the map induced by the inclusions on zeroth homology. It basically works the same way as $H_1(i,j)$, so if you find it necessary to practice these calculations I highly encourage you to work it out that way as well!

(5) Let the topological space M be Hausdorff and locally Euclidean of dimension $d \geqslant 1$ (for example, M could be a manifold).

  (a) Use excision to compute $H_n(M, M\backslash\{x\})$ for any point $x \in M$.

  (b) Consider the case when M is an open Möbius strip, i.e. has no boundary. Pick a generator $\mu_x \in H_n(M, M\backslash\{x\}) \cong \mathbb{Z}$ and describe what happens with μ_x if one walks along the meridian of the Möbius strip.

Solution: Pick an arbitrary point $x \in M$. Since M is locally Euclidean we can find a small neighborhood $U_x \subset M$ for any x such that $U_x \cong \mathbb{R}^d$, where d[3] is the dimension of the space. The subset $M\backslash U_x \subset M\backslash\{x\}$ satisfies the condition for excision (check them!), so we have the following isomorphism of homology groups

$$H_*\big(M, M\backslash\{x\}\big) \cong H_*\Big(M\backslash(M\backslash U_x), (M\backslash\{x\})\backslash(M\backslash U_x)\Big) \cong H_*\big(U_x, U_x\backslash\{x\}\big).$$

By construction we have that U_x is homeomorphic to $\mathbb{R}^d$. In addition $U_x\backslash\{x\} \cong \mathbb{R}^d\backslash\{x\}$ which we have seen in class to deformation retract onto S^{d-1}. Hence, by homotopy invariance,

$$H_n(U_x) \cong \begin{cases} \mathbb{Z}, & n = 0 \\ 0, & \text{else,} \end{cases} \qquad H_n(U_x\backslash\{x\}) \cong \begin{cases} \mathbb{Z}, & n = 0, d-1 \\ 0, & \text{else.} \end{cases}$$

[2]Note that this (fortunately) agrees with what we would have gotten by setting $g = 1$ in the previous exercise. It is a good habit to always double-check your answer when you can!

[3]Note that we changed the notation for the dimension from n to d to avoid a confusion with the indices].

We use the long exact sequence for relative homology for $U_x \backslash \{x\} \subset U_x$.

For $n \geqslant 2$ we find that $H_n(U_x, U_x\backslash\{x\}) \cong H_{n-1}(U_x\backslash\{x\}) \cong H_{n-1}(S^{d-1})$ since the (surrounding) homology groups of U_x are trivial. For the lower degrees we have

$$\begin{tikzcd} 0 & H_0(U_x, U_x\backslash\{x\}) \arrow[l] & H_0(U_x) \arrow[l] \arrow[d, "\simeq"] & H_0(U_x\backslash\{x\}) \arrow[l, "H_0(\iota)"'] \arrow[d, "\simeq"] & H_1(U_x, U_x\backslash\{x\}) \arrow[l] & H_1(U_x) \arrow[l] \arrow[d, "\simeq"] & \cdots \arrow[l] \\ & & \mathbb{Z} & \mathbb{Z} & & 0 & \end{tikzcd}$$

The inclusion $\iota : U_x\backslash\{x\} \hookrightarrow U_x$ induces an isomorphism $H_0(\iota)$ on the zeroth homology groups (Why?). Hence, from exactness of the sequence we learn that $H_1(U_x, U_x\backslash\{x\}) = 0 = H_0(U_x, U_x\backslash\{x\})$. Summarizing, we have

$$H_n(M, M\backslash\{x\}) \cong H_n(U_x, U_x\backslash\{x\}) \cong \begin{cases} \mathbb{Z}, & n = d \\ 0, & \text{else.} \end{cases}$$

For part (b) pick a point $x \in M$ and choose a 2-simplex $\alpha : \Delta^2 \to M$ such that $\partial\alpha \in C_1(M\backslash\{x\})$. In particular this 2-simplex comes with an orientation! Walking along the meridian of the Möbius band reverses the orientation of the 2-simplex (One illustration of this only with a crab instead of a 2-simplex can be found at: Illustration). This shows that the Möbius band is non-orientable.

(6) Prove the following properties of the degree map:

(a) Let $f^{(n)} : S^n \to S^n$ be the map $(x_0, x_1, ..., x_n) \mapsto (-x_0, x_1, ..., x_n)$. Show that $f^{(n)}$ has degree -1.

(b) Show that for $f, g : S^n \to S^n$ we have

$$\deg(F \circ (f \vee g) \circ T) = \deg(f) + \deg(g).$$

Solution: Recall from the computation of the homology groups of spheres (lecture on Tuesday 28.01) that we have isomorphisms

$$H_k(S^n) \xrightarrow{\delta} H_{k_1}(S^{n-1} \times (0,1)) \overset{H_{n-1}(\iota)^{-1}}{\cong} H_{k-1}(S^{n-1}).$$

We denote this composition by D, i.e. $D := H_{n-1}(\iota)^{-1} \circ \delta$.

Let $\mu_0 := [+1] - [-1] \in H_0(S^0)$ and $\mu_1 \in H_1(S^1) \cong \pi_1(S^1)$[4] be the degree one map (i.e. the class of the identity on S^1 which corresponds to the class of the loop $t \mapsto e^{2\pi i t}$). Define the higher μ_n as $D\mu_n = \mu_{n-1}$. Then μ_n is called the *fundamental class* in $H_n(S^n)$.

We prove the claim by induction. First,

$$f^{(0)}(\mu_0) = f^{(0)}([+1] - [-1]) = [-1] - [+1] = -\mu_0.$$

The morphism D is natural, so we have

$$H_n(f^{(n)})\mu_n = H_n(f^{(n)})D^{-1}\mu_{n-1} = D^{-1}H_{n-1}(f^{n-1})\mu_{n-1} \overset{(*)}{=} D^{-1}(-\mu_{n-1}) = -\mu_n,$$

[4] Here the fundamental group is abelian, so in particular it is equal to its abelianization. This isomorphism is not true in general, c.f. Exc 5 Sheet 11.

where step $(*)$ follows from the induction hypothesis. Thus, we can conclude that the degree of $f^{(n)}$ is -1.

For part (b) we note that the map $\widetilde{H}_n(T)$ sends μ_n to $(\mu_n, \mu_n) \in \widetilde{H}_n(S^n) \oplus \widetilde{H}_n(S^n) \cong \widetilde{H}_n(S^n \vee S^n)$, where the isomorphism follows from the fact that our spaces are well-pointed (i.e., they satisfy the conditions for excision). Under the isomorphism the map $\widetilde{H}_n(f \vee g)$ corresponds to $(\mu_n, \mu_n) \mapsto (\widetilde{H}_n(f)\mu_n, \widetilde{H}_n(g)\mu_n)$ and this yields $(\deg(f)\mu_n, \deg(g)\mu_n)$. Under the fold map this is sent to the sum, which is exactly what we wanted to show.

Note that this construction gives a generalization of the additivity relation $\deg(\omega'' \star \omega') = \deg(\omega'') + \deg(\omega')$ which follows from concatenation of paths in the case of S^1.

(7) (a) Let SX denote the suspension of a topological space X. Show by a Mayer-Vietoris sequence that for all n there are isomorphisms $\widetilde{H}_n(SX) \cong \widetilde{H}_{n-1}(X)$.

(b) For $f : S^n \to S^n$ show that suspension leave the degree invariant, i.e.

$$\deg(S(f)) = \deg(f).$$

Conclude that for every integer $k \in \mathbb{Z}$ there is a continuous map $f : S^n \to S^n$ with $\deg(f) = k$. *Hint:* Recall that for $X = S^n$ we have $SX \cong S^{n+1}$.

Solution: Recall that the suspension of a topological space is defined as $SX := X \times [0,1]/(X \times \{0\}, X \times \{1\})$. The suspension of a space can be seen as two cones glued together at their bases. Denoting the upper cone by C_+X and the lower cone by C_-X, we have $SX = C_+X \cup_X C_-X$. Their intersection is homeomorphic to the topological space X. The Mayer-Vietoris sequence becomes

$$\cdots \leftarrow \widetilde{H}_n(C_+X) \oplus \widetilde{H}_n(C_-X) \cong 0 \oplus 0 \leftarrow \widetilde{H}_n(X) \leftarrow \widetilde{H}_{n+1}(SX) \leftarrow 0 \oplus 0 \leftarrow \cdots$$

since cones are contractible and hence have trivial homology groups. Exactness of the sequence then gives the desired isomorphism.

For part (b) we first recall that the suspension of S^n is S^{n+1} which together with the isomorphism from (a) gives $\widetilde{H}_{n+1}(SS^n) \cong \widetilde{H}_{n+1}(S^{n+1}) \cong \widetilde{H}_n(S^n)$.

Note that the isomorphism in (a) comes from the connecting homomorphism δ which in particular is functorial (Proposition 1 in Lecture nr 25). Also, it sends $\mu_{n+1} \in \widetilde{H}_{n+1}(S^{n+1})$ to $\pm\mu_n \in \widetilde{H}_n(S^n)$. We have the commuting diagram

$$\begin{tikzcd}
\widetilde{H}_{n+1}(S^{n+1}) \arrow[r, "\cong"] & \widetilde{H}_{n+1}(SS^n) \arrow[r, "\widetilde{H}_{n+1}(Sf)"] \arrow[d, "\delta"] & \widetilde{H}_{n+1}(SS^n) \arrow[d, "\delta"] & \widetilde{H}_{n+1}(S^{n+1}) \arrow[l, "\cong"'] \\
& \widetilde{H}_n(S^n) \arrow[r, "\widetilde{H}_n(f)"] & \widetilde{H}_n(S^n) &
\end{tikzcd}$$

By tracing $\pm\mu_{n+1} \in \widetilde{H}_{n+1}(SS^n)$ through the diagram we find that $\pm\deg(f)\mu_n = \pm\deg(Sf)\mu_n$, with the same sign. Hence, suspension leaves the degree of the map f invariant. Recall that we have maps of any degree k of S^1 (what are they?), so by using the isomorphism $SS^n \cong S^{n+1}$ and the result of this exercise it follows that the same is true for any S^n as well.

(8) We define the *Euler characteristic* $\chi(X)$ as the alternating sum $\chi(X) := \sum_i (-1)^i \text{Rank}(C_i)$. Show that

$$\chi(X) := \sum_n (-1)^n \operatorname{Rank}(H_n(X))$$

Solution: The proof is exactly the same as that of Theorem 2.44 in Hatcher. Hatcher uses the chain complex from a CW complex structure on the topological space X, but the argument is purely algebraic and works exactly the same way when working with the chain complex coming from a Δ-complex structure on X.

In particular, this means that $\chi(X)$ only depends on the homotopy type of X and is independent of the choice of Δ-structure (or CW-complex structure) on X!

(9) (a) Let X be a path-connected, locally path-connected, and simply connected topological space. Let $p : E \to B$ be a covering with E contractible. Prove that every continuous map $f : X \to B$ induces only zero maps in reduced homology, i.e. for all $n \in \mathbb{N}_0$ we have $\widetilde{H}_n(f) = 0$.

(b) Show that for $n, m \in \mathbb{N}$ with $m \geqslant 2$, any map $\mathbb{S}^m \to \mathbb{T}^n = (\mathbb{S}^1)^n$ induces the zero map on all reduced homology groups. Give a counterexample for $m = 1$.

Solution: Here, for every continuous map $f : X \to B$ there exists a map $\widetilde{f}$ (why?) such that the below diagram commutes

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, dashed, "\widetilde{f}"] \arrow[r, "f"'] & B. \end{tikzcd}$$

Since E is contractible we know that $H_n(E) = 0$ for all $n > 0$, which implies that $H_n(p) = 0$. On the level of homology we have $H_n(f) = H_n(p \circ \widetilde{f}) = H_n(p) \circ H_n(\widetilde{f})$ since H_* is functorial. Hence $H_n(f)$ must also be trivial for all $n > 0$. Finally, for $n = 0$ we simply use that X is path-connected so $\widetilde{H}_0(X) = 0$ which ensures that $\widetilde{H}_0(f) = 0$ as well.

In part (b) we note that when $m \geqslant 2$ the sphere S^m is path-connected, locally path-connected and simply connected. Also, T^n has the covering $p : \mathbb{R}^n \to T^n$ given by $(\phi_1, ..., \phi_n) \mapsto (e^{i\pi\phi_1}, ..., e^{i\pi\phi_n})$. We know that $\mathbb{R}^n$ is contractible, so for this situation the result in part (a) applies.

For the case $m = 1$ we can e.g. consider the identity map $\text{id} : S^1 \to T^1$ which gives $\widetilde{H}_1(\text{id}) = H_1(\text{id}) = \text{id}_{\mu_1}$. This is non-trivial and hence gives a counterexample.