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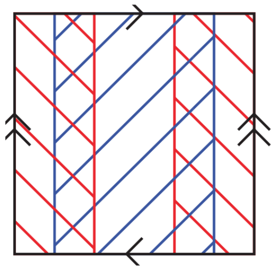
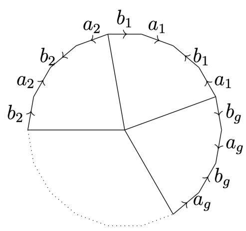
Algebraic Topology – Exercise 13

(1) Compute the homology of $\mathbb{R}^3 \backslash A$, where A is the upper hemisphere of the unit sphere S^2 in $\mathbb{R}^3$.

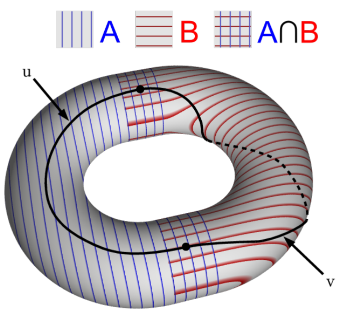
(2) Cover a Klein bottle by two open subsets that are each homeomorphic to an open Möbius strip (see the parts shaded in blue and red in the figure). Compute the reduced homology of the Klein bottle with the help of this cover. *Hint:* Start with constructing the reduced version of the Mayer-Vietoris sequence.



(3) Let F_g denote the closed oriented surface of genus $g \in \mathbb{N}_0$, obtained by identifying edges with the same label according to the orientations indicated in the $4g$-gon in the below Figure. Use the Mayer-Vietoris Theorem to compute $H_*(F_g)$. (Do not use the Hurewicz Theorem to compute H_1)!

$F_g :=$ 

(4) Compute the homology groups of the torus T. *Hint:* Use Mayer-Vietors, for example with two overlapping cylinders as indicated in the below Figure.



(5) Let the topological space M be Hausdorff and locally Euclidean of dimension $n \geqslant 1$ (for example, M could be a manifold).

(a) Use excision to compute $H_n(M, M\backslash\{x\})$ for any point $x \in M$. These groups are sometimes referred to as the *local homology groups* of M at $x \in M$.

(b) Consider the case when M is an open Möbius strip, i.e. has no boundary. Pick a generator $\mu_x \in H_n(M, M\backslash\{x\}) \cong \mathbb{Z}$ and describe what happens with μ_x if one walks along the meridian of the Möbius strip.

Definition. Consider a map $f : S^n \to S^n$ with $n > 0$. The induced map $f_* : H_n(S^n) \to H_n(S^n)$ is a homomorphism from $\mathbb{Z}$ to itself and thus must be of the form $f_*(\alpha) = d\alpha$ for some integer d depending only on the map f. This integer is called the *degree of* f and is denoted $\deg f$.

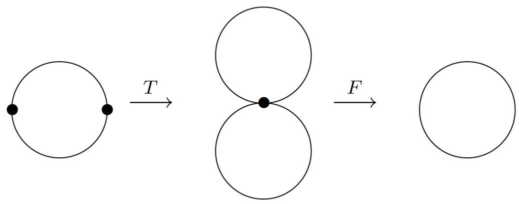
(6) Prove the following properties of the degree map:

(a) Let $f^{(n)} : S^n \to S^n$ be the map

$$(x_0, x_1, ..., x_n) \mapsto (-x_0, x_1, ..., x_n).$$

Show that $f^{(n)}$ has degree -1.

(b) Consider the *pinch map* $T : S^n \to S^n/S^{n-1} \cong S^n \vee S^n$ and the *fold map* $F : S^n \vee S^n \to S^n$, which is induced by the identity of S^n. The figure below depicts the two maps for $n = 1$.



(i) Illustrate the pinch and fold map for $n = 2$.

(ii) Show that for $f, g : S^n \to S^n$ we have

$$\deg(F \circ (f \vee g) \circ T) = \deg(f) + \deg(g).$$

(7) (a) Let SX denote the suspension of a topological space X. Show by a Mayer-Vietoris sequence that for all n there are isomorphisms $\widetilde{H}_n(SX) \cong \widetilde{H}_{n-1}(X)$.

(b) For $f : S^n \to S^n$ show that suspension leave the degree invariant, i.e.

$$\deg(S(f)) = \deg(f).$$

Conclude that for every integer $k \in \mathbb{Z}$ there is a continuous map $f : S^n \to S^n$ with $\deg(f) = k$. *Hint:* Recall that for $X = S^n$ we have $SX \cong S^{n+1}$.

(8) Let X be a Δ-complex of dimension n (i.e. X does not have any s-simplicies for $s > n$) and let $C_*(X)$ denote the corresponding chain complex. We define the *Euler characteristic* $\chi(X)$ as the alternating sum $\chi(X) := \sum_i (-1)^i \operatorname{Rank}(C_i(X))$. Show that

$$\chi(X) = \sum_n (-1)^n \operatorname{Rank}(H_n(X))$$

Note that this is a generalization of the formula $\chi(X) = V - E + F$ obtained for polyhedron (i.e. Δ-complexes of dimension 2) in Exercise 6 on Sheet 8.

(9) (a) Let X be a path-connected, locally path-connected, and simply connected topological space. Let $p : E \to B$ be a covering with E contractible. Prove that every continuous map $f : X \to B$ induces only zero maps in reduced homology, i.e. for all $n \in \mathbb{N}_0$ we have $\widetilde{H}_n(f) = 0$.

(b) Show that for $n, m \in \mathbb{N}$ with $m \geqslant 2$, any map $\mathbb{S}^m \to \mathbb{T}^n = (\mathbb{S}^1)^n$ induces the zero map on all reduced homology groups. Give a counterexample for $m = 1$.