Combinatorics of Coxeter Groups
Exercises and reading assignment for June 15th

Read sections 4.1 and 4.2. While reading, keep the example of the symmetric group S_3 in mind. We have $S = \{s_1, s_2\}$ with $m(s_1, s_2) = 3$, so we may choose $k_{s_1,s_2} = k_{s_2,s_1} = 1$. As everything is two-dimensional, you are invited to draw pictures. Thereby, you solve Exercise 1.

Exercise 1

Describe the standard geometric representation of the symmetric group S_3.

Exercise 2

Recall that the Coxeter group (W, S) is called *irreducible* iff the associated Coxeter graph is connected. This means that for each $s, s' \in S$, there is a chain $s = s_1, \ldots, s_q = s' \in S$ such that $m(s_i, s_{i+1}) \neq 2$ for $i = 1, \ldots, q-1$.

Consider a geometric representation $\sigma : W \to \mathrm{GL}(V)$ as in section 4.2. Recall that an *invariant subspace* is a linear subspace $U \subseteq V$ such that $\sigma(w)(U) = U$ for all $w \in W$. Finally recall that the representation (V, σ) is called *irreducible* if the only invariant subspaces are $\{0\} \subseteq V$ and V itself.

Show that (W, S) is irreducible iff (V, σ) is irreducible.[1]

[1] If $S = \emptyset$, both (W, S) and (V, σ) are defined to be not irreducible. You may assume $S \neq \emptyset$.