1. Answers to Emails
2. Questions for further discussions

Automorphisms

Let (W, S) be a Coxeter system.

- A *Coxeter group automorphism* is a group automorphism $\varphi : W \to W$ such that $\varphi(S) = S$.
- A *Coxeter graph automorphism* is a bijection $\varphi : S \to S$ such that $m(s, s') = m(\varphi(s), \varphi(s'))$ for all $s, s' \in S$.

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It follows from the definition of Coxeter groups that

$$\{\text{Coxeter group automorphisms}\} \to \{\text{Coxeter graph automorphisms}\}$$

$$\varphi \mapsto \varphi|_S$$

is a bijection.

Bruhat order automorphisms, case $|S| \geq 3$

Each automorphism of $(W, \leq)$ must induce a permutation of S. By Theorem 2.3.5, we get a surjective group homomorphism

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whose kernel is generated by the automorphism $\varphi(x) := x^{-1}$. Thus, the group of Bruhat order automorphisms decomposes as a direct product

$$(\mathbb{Z}/2\mathbb{Z}) \times \{\text{Coxeter graph automorphisms}\}.$$

Bruhat order automorphisms, case $|S| = 2$

The dihedral group $I_2(m) = a \overset{m}{\text{---}} b$ is given as follows: There is one element of length 0 (namely the identity) and one of length m (namely $w_0 = (abab\ldots)_m = (baba\ldots)_m$). For any $1 \leq k \leq m-1$, there are exactly two elements of length k, namely $(abab\ldots)_k$ and $(baba\ldots)_k$.

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The Bruhat order is given by

$$w_1 \leq w_2 \iff (w_1 = w_2 \text{ or } \ell(w_1) < \ell(w_2)).$$

We see that the Bruhat order automorphisms are precisely the length-preserving maps $W \to W$, so the group is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{m-1}$ (Exercise 2.2).

Bruhat order anti-automorphisms

Let A_1 be the set of Bruhat order automorphisms, A_2 be the set of Bruhat order anti-automorphisms and $A_3 = A_1 \cup A_2$. Then A_1 and A_3 are groups with respect to function composition. Now if ψ is any anti-automorphism of the Bruhat order, e.g. $\psi(x) = xw_0$, then

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Thus the set of Bruhat order anti-automorphisms is

- empty, if W is infinite (as there is a smallest, but no largest element) or
- in bijection with the set of Bruhat order automorphisms, otherwise.

Weak order automorphisms

If (W, S) is chosen somewhat reasonably (it suffices to require that the ∞-labeled edges are pairwise disjoint), then

$$\{\text{Weak order automorphisms}\} \to \{\text{Coxeter graph automorphisms}\}$$
$$\varphi \mapsto \varphi|_S$$

is a bijection (Theorem 3.2.5).

Geometry of root systems

For the symmetric groups, the root system can be identified with

$$\Phi^+ = \{e_i - e_j \mid i < j\} \subseteq V := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 + \cdots + x_n = 0\}.$$

For the dihedral group $I_2(m)$, the root system consists of $2m$ evenly spaced vectors on the unit circle of $\mathbb{R}^2$.

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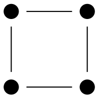
Other than that, there aren't many intuitive descriptions, so one should work with the formula

$$t_\gamma(\beta) = \beta - 2(\gamma \mid \beta)\gamma$$

for $\beta, \gamma \in \Phi$.

The basics

Let (W, S) be a Coxeter group of type $\tilde{A}_3$, so its Coxeter graph is given as follows:



- What does that mean precisely? Can you define (W, S) in group theoretic terms?
- How many elements in W are there of length 0, resp. 1 or 2?

Exchange property

Let $W = (\mathbb{Z}/2)^2$ be the Klein four group, and $S := W \setminus \{1\}$.
Show that (W, S) is *not* a Coxeter system.

Bruhat and weak order

Give an example of a Coxeter system (W, S) with two elements w_1, w_2 such that $w_1 \leq w_2$ holds in the Bruhat order, but $w_1 \nleq_R w_2$ in the right weak order.

Parabolic subgroups

Let (W, S) be a Coxeter system with W finite, and let $J \subseteq S$ be any subset. Show that the order of W^J divides the order of W.

Geometric representation

Let (W, S) be a Coxeter system of type B_3.

- Write down the Coxeter graph.
- How many elements does W have?
- Write down the matrices corresponding to the simple reflections for the standard geometric representation.