Reading course: Coxeter groups

Felix Schremmer

April 13, 2021

1 About the course

2 Introduction to Coxeter groups

3 Reading assignment for next week

Structure of the course

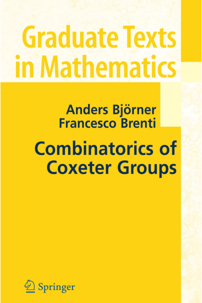
1. I assign reading material and exercises
2. You read the material and try to solve the exercises
3. We discuss your questions and problems

All infos on the course website:

`https://www.groups.ma.tum.de/en/algebra/felix-schremmer/reading-course-coxeter-groups/`

Timeplan

- Today: Introduction to the topic
- Next 9 timeslots: Working through the first chapters of *Combinatorics of Coxeter Groups*.
- Remaining 4 timeslots: An application, precise topic tbd
- Oral exam



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The symmetric group

Definition

Let $n \geq 1$. The *symmetric group* S_n consists of all bijective maps

$$f : \{1, \ldots, n\} \to \{1, \ldots, n\}.$$

Group multiplication is function composition.

Special kinds of elements of S_n are the *transpositions* $s_{i,j}$ for $i, j \in \{1, \ldots, n\}$, defined by

$$s_{i,j}(i) = j, \qquad s_{i,j}(j) = i, \qquad s_{i,j}(k) = k \quad \forall k \neq i, j.$$

The elements $s_{i,i+1}$ for $i = 1, \ldots, n-1$ are called *standard transpositions*.

A lemma

Lemma

The symmetric group S_n is generated by the standard transpositions $s_{i,i+1},\ i = 1, \ldots, n-1$.

We want to give a geometric proof of this lemma.

A group representation

Denote by V either the vector space $\mathbb{R}^n$ or

$$\{v \in \mathbb{R}^n \mid v_1 + v_2 + \cdots + v_{n-1} + v_n = 0\}.$$

Then S_n acts on V by permutation of coordinates:

$$fv = (v_{f^{-1}(1)}, v_{f^{-1}(2)}, \ldots, v_{f^{-1}(n-1)}, v_{f^{-1}(n)}).$$

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We note that $s_{i,j}v = v \iff v_i = v_j$. If $i \neq j$, these form a hyperplane

$$H_{i,j} = \{v \in V \mid v_i = v_j\}.$$

Weyl chambers

Call a vector $v \in V$ *regular* if $v_i \neq v_j$ for $i \neq j$. i.e.

$$V^{\mathrm{reg}} := V \setminus \bigcup_{i \neq j} H_{i,j}.$$

A *Weyl chamber* is a connected component of V^{reg}. e.g.

$$C_0 = \{v \in V \mid v_1 > v_2 > \cdots > v_{n-1} > v_n\}.$$

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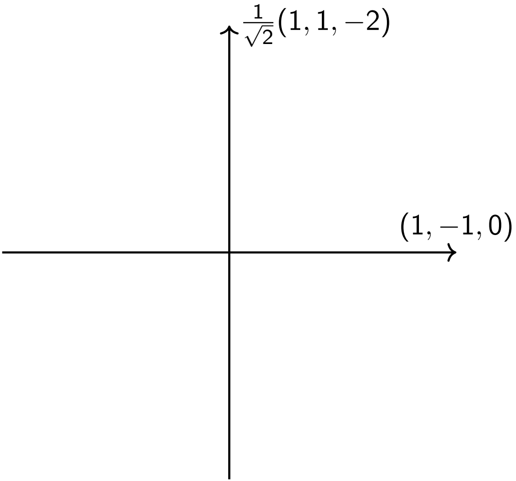
Then there is a 1-1 correspondence

$$\begin{array}{ccc} S_n & \leftrightarrow & \text{weyl chambers} \\ f & \mapsto & fC_0. \end{array}$$

Weyl chambers of S_3

$$V = \{(v_1, v_2, v_3) \in \mathbb{R}^3 \mid v_1 + v_2 + v_3 = 0\}.$$



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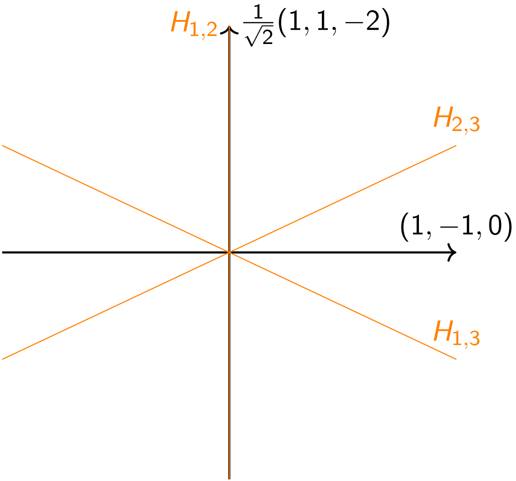

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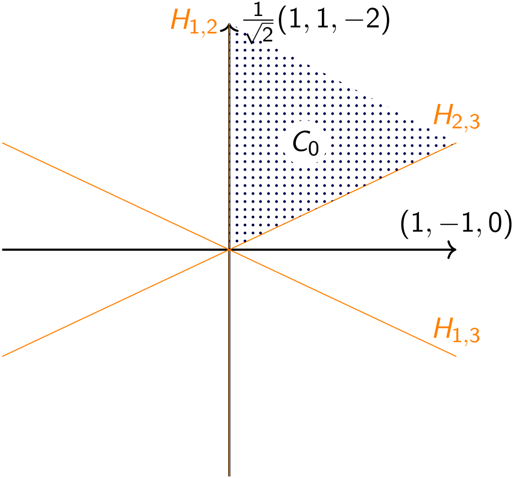

Length

For $f \in S_n$, denote

$$\ell(f) := \#\{H_{i,j} \mid H_{i,j} \text{ lies in between } C_0 \text{ and } fC_0\}.$$

The hyperplanes neighbouring C_0 are precisely the hyperplanes of the form $H_{i,i+1}$.

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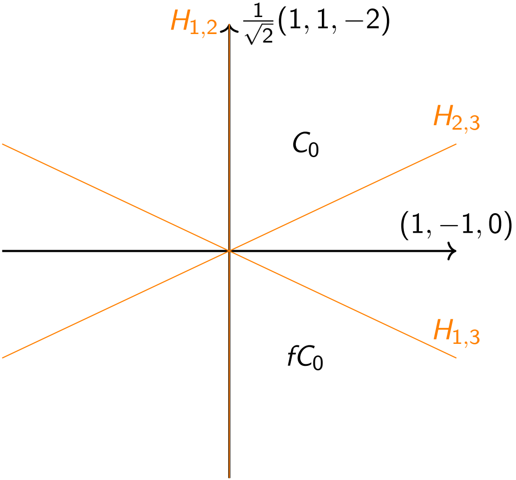
Now if $H_{i,i+1}$ lies in between C_0 and fC_0, then

$$\begin{aligned}\ell(s_{i,i+1}f) &= \#\{H_{i,j} \mid H_{i,j} \text{ lies in between } s_{i,i+1}C_0 \text{ and } fC_0\} \\ &= \#\{H_{i,j} \neq H_{i,i+1} \mid H_{i,j} \text{ lies in between } C_0 \text{ and } fC_0\} \\ &= \ell(f) - 1.\end{aligned}$$

Illustration for S_3



$f \in S_3$
$f(1) = 3$
$f(2) = 1$
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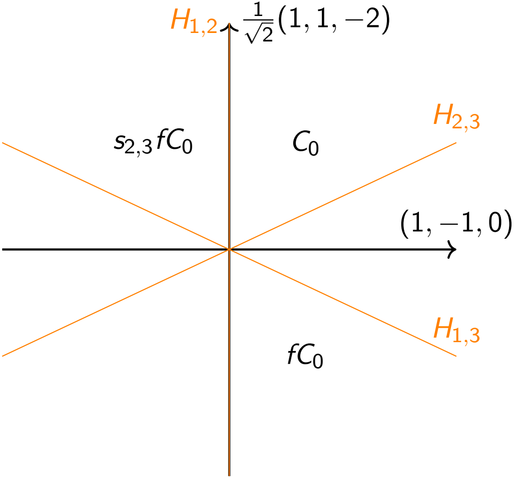

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Algebraic meaning of length

Proposition

Each $f \in S_n$ can be written as a product of $\ell(f)$ standard transpositions $s_{i,i+1}$, but not as a product of less than $\ell(f)$ standard transpositions.

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Proof.

As long as $\ell(f) > 0$, we can write $f = s_{i,i+1}(s_{i,i+1}f)$ with $\ell(s_{i,i+1}f) = \ell(f) - 1$. Iterate this $\ell(f)$ times.
For the converse, we can show $\ell(s_{i,i+1}f) \leq \ell(f) + 1$ for all $f \in S_n$ similar to the previous argument. □

Towards Coxeter groups

We saw that S_n is generated by the set

$$S = \{s_{1,2}, s_{2,3}, \ldots, s_{n-2,n-1}, s_{n-1,n}\}.$$

Moreover, writing $s_i = s_{i,i+1}$, we have the relations

$$s_i^2 = 1, \qquad s_i s_j = s_j s_i \text{ if } |i-j| \geq 2,$$
$$(s_i s_{i+1})^3 = 1.$$

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Every other equation consisting of just s_i's can be derived from the above relations and the laws of group theory.

Coxeter groups

Definition

Let S be a set and $m : S \times S \to \{1, 2, 3, \ldots, \infty\}$ a function such that

$$m(s, s') = m(s', s) \text{ and } m(s, s') = 1 \iff s = s'.$$

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The *Coxeter group* associated to this is the group W presented as follows:

Generators: The set S.

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In the case of S_n: $S = \{s_1, \ldots, s_{n-1}\}$.

$$m(s_i, s_j) = \begin{cases} 1, & i = j, \\ 3, & |i - j| = 1, \\ 2, & |i - j| \geq 2. \end{cases}$$

Words in S

Let S, m be as in the previous definition. The set of *words* is

$$S^* := \bigcup_{m \geq 0} S^m = \{(s_1, \ldots, s_m) \mid m \geq 0, s_i \in S\}.$$

Let $w, w' \in S^*$. We say w' *is an elementary reduction of* w if w' is obtained from w by deleting a subword of the form

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We write $w \sim w'$ iff there exists words

$$w = w_1, w_2, \dots, w_{q-1}, w_q = w' \in S^*$$

such that each w_i is an elementary reduction of w_{i+1}, or vice versa.

An explicit description of W

We define $W = {}^{S^*}\!/_{\sim}$ to be the set of equivalence classes of words. If $[w]_\sim, [w']_\sim \in W$, let $w \circ w'$ be the composed word in S^* and

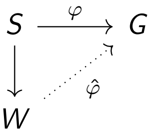
$$[w]_\sim \cdot [w']_\sim := [w \circ w']_\sim.$$

This gives W the structure of a group.

A universal property

W is equivalently characterized by the following properties:

1. We have a map $S \to W$ such that for all $s, s' \in S$ with $m(s, s') \neq \infty$:
$$(ss')^{m(s,s')} = 1.$$
2. For each group G and each mapping $\varphi : S \to G$ satisfying the condition
$$(\varphi(s)\varphi(s'))^{m(s,s')} = 1 \in G \text{ if } s, s' \in S \text{ with } m(s, s') \neq \infty,$$
there exists a unique group homomorphism $\hat{\varphi} : W \to G$ with $\hat{\varphi}|_S = \varphi$.



Why study Coxeter groups?

Consider the group GL_n of invertible $n \times n$-matrices. Let $B \subseteq \mathrm{GL}_n$ be the subgroup of upper triangular matrices and $S_n \subseteq \mathrm{GL}_n$ the set of permutation matrices. Then

$$\mathrm{GL}_n = BS_nB.$$

This is the starting point to *understanding the infinite group* GL_n *by studying the finite group* S_n.

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Other reasons:

- Geometry of reflections and the groups generated by them.
- Knot theory, i.e. studying embeddings $S^1 \hookrightarrow \mathbb{R}^3$.
- ...

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Combinatorics of Coxeter Groups

- 1.1 Coxeter systems
- 1.2 Examples (leave out/skim only the examples 1.2.10 and 1.2.11).
- 1.3 A permutation representation
- 1.4 Reduced words and the exchange property
- Exercises 2, 8 and 10 of Chapter 1