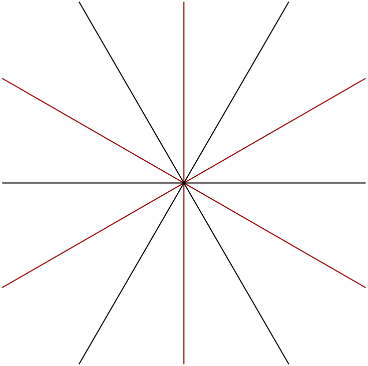
Exercise 2



D_{12} is the generated by the six reflections. It is the direct product of S_3 (generated by the red reflections) and $\mathbb{Z}/2\mathbb{Z}$ (generated by the point reflection).

Exercise 2, alternative solution

The coxeter groups $\bullet\overset{6}{-}\bullet$ and $\bullet-\bullet \quad \bullet$ both have

- twelve elements,
- a unique subgroup of order 3,
- a subgroup isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$,
- a pair of elements a, b with $ab \neq ba$.

Conclude: Both are a non-abelian semi-direct product $\mathbb{Z}/3\mathbb{Z} \ltimes (\mathbb{Z}/2 \times \mathbb{Z}/2)$, for which there is only one possibility.

Exercise 8

Induction by k. The statement is obvious for $k = 0, 1$.
If $s_1 \cdots s_k$ satisfies that

$$s_1 \cdots s_{i-1} s_i s_{i-1} \cdots s_1, \qquad i = 1, \ldots, k$$

are pairwise distinct, the same holds for $s_2 \cdots s_k$.
By induction, $s_2 \cdots s_k$ is reduced and

$$s_1 \neq s_1 \cdots s_{i-1} s_i s_{i-1} \cdots s_1 \implies s_1 \neq s_2 \cdots s_{i-1} s_i s_{i-1} \cdots s_2$$

for $i = 2, \ldots, k$. Thus $s_1 \cdots s_k$ is reduced.

Exercise 10

Let $t = s_1 \cdots s_k$ be a reduced decomposition, so k is odd. By Theorem 1.3.2 (ii), we must have $n(s_1 \cdots s_k; t)$ odd. In particular, there exists $i \in \{1, \dots, k\}$ with

$$t = s_1 s_2 \cdots s_{i-1} s_i s_{i-1} \cdots s_2 s_1.$$

If $i \leq \frac{k+1}{2}$, this palindromic expression has $2i - 1 \leq k = \ell(t)$ factors, so we are done.
If $i \geq \frac{k+1}{2}$, use

$$\begin{aligned} t = t^{-1} t t =& (s_k \cdots s_1)(s_1 \cdots s_{i-1} s_i s_{i-1} \cdots s_1)(s_1 \cdots s_k) \\ =& \underbrace{s_k \cdots s_{i+1} s_i s_{i+1} \cdots s_k}_{2(k-i)+1 \leq k = \ell(t) \text{ factors}}. \end{aligned}$$