On the Exchange Property

Correct Exchange Property: Let $w = s_1 \cdots s_k$ be reduced, $s \in S$ and $\ell(sw) \leq \ell(w)$. Then $sw = s_1 \cdots \widehat{s_i} \cdots s_k$ for some i.

Wrong Exchange Property: Let $w = s_1 \cdots s_k$ be reduced, $s \in S$ and $\ell(sw) < \ell(w)$. Then $sw = s_1 \cdots \widehat{s_i} \cdots s_k$ for some i.

If (W, S) satisfies the Wrong Exchange Property, it might not be a Coxeter system. e.g. $W = V_4 = (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ and $S = W \setminus \{1\}$.

The above Exchange Properties are of course equivalent if we moreover assume $\ell(sw) \neq \ell(w)$ for all $s \in S, w \in W$.

Exercise 1.18

Recall

$$\Sigma(W') = \{t \in T \cap W' \mid \ell_{(W,S)}(t't) > \ell_{(W,S)}(t) \quad \forall t \neq t' \in T \cap W'\}.$$

Let $w = t_1 \cdots t_k$ be a reduced expression for $(W', \Sigma(W'))$ and $t \in \Sigma(W')$.

Case 1. $\ell_{(W,S)}(tw) < \ell_{(W,S)}(w)$: By the Exchange Property, we find some t_i with reduced expression $t_i = s_1 \cdots s_q$ for (W, S) such that

$$\begin{aligned}
& tw = t_1 \cdots t_{i-1} s_1 \cdots \widehat{s_j} \cdots s_q t_{i+1} \cdots t_k \\
\iff & s_1 \cdots \widehat{s_j} \cdots s_q = \underbrace{t_{i-1} \cdots t_1 t t_1 \cdots t_{i-1}}_{=: t'} t_i. \\
\implies & t' \in T \cap W' \text{ and } \ell_{(W,S)}(t' t_i) < \ell_{(W,S)}(t_i) \implies t_i = t'. \\
\implies & tw = t_1 \cdots \widehat{t_i} \cdots t_k.
\end{aligned}$$

Exercise 1.18, continued

Case 2. $\ell_{(W,S)}(tw) > \ell_{(W,S)}(w)$. Then Case 1 shows $\ell_{(W',\Sigma(W'))}(tw) > \ell_{(W',\Sigma(W'))}(w)$.

Upshot. If $\ell_{(W',\Sigma(W'))}(tw) \leq \ell_{(W',\Sigma(W'))}(w)$, then we must have

$$tw = t_1 \cdots \widehat{t_i} \cdots t_k$$

for some $i \in \{1, \ldots, k\}$. This verifies the Exchange Property for $(W', \Sigma(W'))$.

Exercise 1.18, Addendum

It remains to show that W' is generated by $\Sigma(W')$. Suppose there is $t \in T \cap W'$ which is not in the subgroup generated by $\Sigma(W')$, and pick one such t with $\ell_{(W,S)}(t)$ minimal.
Fix a palindromic reduced expression (cf. last week)

$$t = s_1 \cdots s_k, \qquad s_i = s_{k+1-i}.$$

Since $t \notin \Sigma(W')$, we find $t \neq t' \in T \cap W'$ with $\ell(t't) < \ell(t)$. By the Exchange Property, $t't = s_1 \cdots \widehat{s_i} \cdots s_k$. Up to replacing t' by $tt't$, we may assume $i \leq \frac{k+1}{2}$.
If $i = \frac{k+1}{2}$, we get $tt' = 1$, so $t = t'$, contradiction. Thus, $i < \frac{k+1}{2}$. Then

$$\ell(t') = \ell(s_1 \cdots s_{i-1} s_i s_{i-1} \cdots s_1) \leq 2i - 1 < k.$$
$$\ell(t'tt') = \ell(s_1 \cdots \widehat{s_i} \cdots \widehat{s_{k+1-i}} \cdots s_k) \leq k - 2 < k.$$

By assumption, t' and $t'tt'$ are in the subgroup generated by $\Sigma(W')$, hence so is t. Contradiction!

Exercise 2.3

$S = J_1 \dot\cup J_2 \dot\cup \cdots \dot\cup J_k$ such that $ss' = s's$ whenever $s \in J_i, s' \in J_j$ and $i \neq j$.
W is the product of the W_{J_i} (the subgroups generated by the J_i):
Whenever $w = s_1 \cdots s_q \in W$, rearrange the terms such that

$$w = \underbrace{s_1 \cdots s_{j_1}}_{\in W_{J_1}} \underbrace{s_{j_1+1} \cdots s_{j_2}}_{\in W_{J_2}} \cdots \underbrace{s_{j_{k-1}} \cdots s_q}_{\in W_{J_k}}$$
$$=: w_1 \cdots w_k.$$

Then $T_R(w) = T_R(w_1) \dot\cup \cdots \dot\cup T_R(w_k)$, so the $w_1, \ldots, w_k$ are uniquely determined.
If $w = w_1 \cdots w_k \xrightarrow{t} w' = w_1' \cdots w_k'$, we have $t \in W_{J_i}$ for some $i \in \{1, \ldots, k\}$. Then $w_j = w_j'$ for $j \neq j'$ and $w_i \xrightarrow{t} w_i'$. Like this, we see that the Bruhat order on W is the product of the Bruhat orders of the W_{J_i}.

Bruhat Cells for GL_3

It is a fact that $\mathrm{GL}_3 = \dot{\bigcup}_{w \in S_3} BwB$, where B is the set of upper triangular matrices. Now consider

$$A = \begin{pmatrix} * & * & * \\ a & b & * \\ c & d & * \end{pmatrix} \in \mathrm{GL}_3(\mathbb{C}).$$

$w \in S_3$	Condition for $A \in BwB$	$\dim(BwB/B)$
123	$a = c = d = 0$	0
213	$a \neq 0 = c = d$	1
132	$d \neq 0 = c = a$	1
231	$a, d \neq 0 = c$	2
312	$c \neq 0 = ad - bc$	2
321	$c \neq 0 \neq ad - bc$	3