winter semester 2019/20 Claudia Scheimbauer

Algebraic Topology – rough list of topics covered

The number is the number of the lecture.

1. Basic notions from point-set topology

(1) definition topological space, open, closed sets, neighborhood

(2) continuous maps, subspace topology and its universal property, homeomorphism, definition of category

(3) universal constructions of topological spaces (subspace topology, product topology, quotient topology)

(4) universal constructions continued (quotient topology, coproduct/sum/disjoint union, pushout) Examples: attaching an n-cell, cell complex

(5) cell complexes continued, operations on topological spaces: cone, suspension, wedge sum, smash product; connectedness, path-connectedness, connected components and path components;[1, 2, 3]

(6) $\pi_0(-)$ as a functor, $\pi_0(-)$ is the left adjoint of the inclusion of discrete spaces, computing π_0, locally path-connected, Hausdorff, compact; [3]

(7) Some theorems about compactness, locally compact, Tychonoff's theorem (proof for finite products), Heine-Borel, proper maps, fiber bundles

(8) proper fiber bundles, Hopf fiber bundle

Further reading: basis for topology, closure, boundary

2. Fundamental group

(8) Introduction of what is to come, homotopies relative to a subspace and path homotopies, homotopy as an equivalence relation, homotopy equivalence, deformation retract, contractible [1, 3]

(9) homotopy category, definition fundamental groupoid

(10) first properties of fundamental group(oid), definition of natural transformation, simply connected, statement of Seifert-van Kampen Theorem for groups

(11) computation of $\pi_1(\mathbb{RP}^2, x)$ via Seifert-van Kampen theorem, statement of Seifert-van Kampen Theorem for groupoids, more details on pushouts, pushout of diagram of groupoids of the form BG, first part of proof of of Seifert-van Kampen Theorem for groupoids

(12) proof of of Seifert-van Kampen Theorem for groupoids, Seifert-van Kampen Theorem for $\Pi_{\leq 1}(X, A)$ using that retracts of pushouts are pushouts, example: S^1

(13) Computing $\Pi_{\leq 1}(S^1, \{-1, 1\})$ and $\pi_1(S^1, 1)$, proof of Seifert-van Kampen Theorem for groups, Corollaries of $\pi_1(S^1, 1) \cong \mathbb{Z}$: Fundamental theorem of Algebra, Brouwer's Fixed Point Theorem, Borsuk-Ulam Theorem

(14) Covering spaces: definition and some examples, Lemma on path lifting

(15) Theorem of homotopy lifting, coverings give subgroups of the fundamental group of the base, monodromy functor

(16) lifting criterion via inclusion of image of fundamental group, deck transformations as automorphisms in category of covering spaces, relation between deck transformations and fundamental group of base, universal covering space, existence of universal covering space and Galois correspondence for covering spaces

3. Classification of surfaces

(17) definition of manifold and surface, surfaces as quotients of polygons, examples of surfaces (possibly non-compact) from gluing pattern/labelling scheme: cylinder, Moebius band, torus, Klein bottle, projective plane, double torus, sphere, proper labelling scheme, allowed operations, n-fold dunce cap as non-example of proper labelling scheme

(18) Proof of classification of surfaces from labelling schemes cf. [2]

(19) Fundamental group of genus g surface and its abelianization, Corollary: genus g surfaces and k-fold projective planes are pairwise non-homeomorphic, triangulations of surfaces, classification of surfaces, motivation for first homology as abelianization of fundamental group

4. Homology and some simplicial sets

Mostly following [1]

(20) semi-simplicial complexes (Δ-complexes): simplices, faces and boundary of a simplex, example of torus and Klein bottle with face maps, examples: triangualtions and CW-complexes, reformulation as semi-simplicial set (intuition)

(21) recollection of Δ-complexes, reformulation as semi-simplicial set continued using the categories Δ and Δ_{inj}, geometric realization, free abelian groups, chain complexes, homology groups, examples, definition of simplicial homology

(22) computation of nth simplicial homology of S^n, singular semi-simplicial set Sing and singular homology, morphisms of semi-simplicial complexes, singular homology of $\emptyset$, a point, morphisms of semi-simplicial sets, functoriality of Sing

(23) morphisms of graded abelian groups, chain maps, examples of 1-cycles, functoriality of (singular) homology, computation of H_0 and augmentation, homology of coproducts, chain homotopies induce isomorphisms on homology

(24) homotopies of topological spaces induce chain homotopies using prism operators

(25) (short) exact sequences, snake lemma for connecting homomorphism, long exact sequence of homology groups, relative singular chain complex and relative singular homology, long exact sequence of singular homology

(26) long exact sequence of singular homology: variants for relative homology and reduced homology, computation for two points in C^*, Eilenberg-Steenrod axioms, Excision, statement of locality, Mayer-Vietoris, example: homology of S^n

(27) proof of excision assuming locality and 5-lemma, equivalence of singular and simplicial homology

(28) strong deformation retracts and good pairs, relative homology for good pairs is reduced homology of the quotient, degree of map of S^n, (open subsets of) $\mathbb{R}^n$ and $\mathbb{R}^k$ are not homeomorphic if $n \neq k$, Hairy Ball Theorem

References

[1] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002. available at `https://pi.math.cornell.edu/~hatcher/AT/AT.pdf`.

[2] James R. Munkres. *Topology*. Prentice Hall, Inc., Upper Saddle River, NJ, 2000. Second edition of [MR0464128].

[3] Gerd Laures and Markus Szymik. *Grundkurs Topologie*. Springer-Lehrbuch. [Springer Textbook]. Springer Spektrum, Berlin, revised edition, 2015.

[4] William Fulton. *Algebraic topology*, volume 153 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. A first course.

[5] Klaus Jänich. *Topologie*. Springer-Verlag, Berlin, eighth edition, 2005.

[6] Lynn Arthur Steen and J. Arthur Seebach, Jr. *Counterexamples in topology*. Dover Publications, Inc., Mineola, NY, 1995. Reprint of the second (1978) edition.