Examples in Algebraic Geometry – introduction

Felix Schremmer

September 23, 2021

Timeplan

- Today: introduction and organization
- During the semester: Each of you will give one talk.
- We will decide the details of our weekly meetings in a minute.

The process

- Read and understand your section. Prepare your examples and your talk.
- ≥ 2 weeks before your talk: Send a draft of your talk to prof. Viehmann or me, and arrange a personal meeting
- ≥ 1 week(s) before your talk: One of us discusses your draft with you. This meeting is compulsory.
- Give your talk, present your examples, enlighten your audience.
- After your talk, you get personal feedback.

When and where

- We have the choice between holding the seminar online or in presence.
- We have a lot of freedom choosing the timeslot.
- What are your needs/preferences regarding these choices?

Regarding your talks

- Each talk should take roughly 90 minutes. The n-th talk will happen most likely in the n-th week of the semester.
- Follow the section of Harris' textbook or find your own source.
- Please attend all talks as a courtesy to your fellow students.
- Examples are important in this seminar. Prepare them thoroughly and spend some time elaborating them in the talk
- You are always invited to ask questions to us.
- During your talk, indicate when you would prefer the audience to ask their questions.

A remark on base fields

- Algebraic geometry (for us) happens over a field K.
- For calculations/proofs, it is convenient if K is algebraically closed.
- For intuition/pictures, we will assume $K = \mathbb{R}$.

Affine space

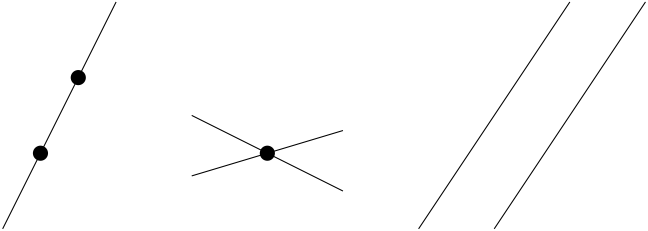
The affine n-space $\mathbb{A}^n$ is the vector space K^n, except that the origin $0 \in K^n$ is no longer distinguished.
Those are the pictures over $\mathbb{R}$

- $\mathbb{A}^0$: Point
- $\mathbb{A}^1$: Line, infinite in both directions
- $\mathbb{A}^2$: Plane
- ...

Shortcomings of the affine spaces I

Consider the affine plane $\mathbb{A}^2$.

- Any two distinct points uniquely determine a line.
- Any two distinct lines uniquely determine a point, *unless they are parallel*.



Idea: Artificially add "intersection points" of parallel lines to $\mathbb{A}^2$.

Shortcomings of the affine spaces II

Perspective drawing can be modelled as a linear map

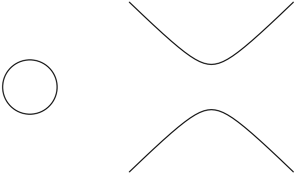
$\mathbb{R}^3 \to \mathbb{R}^2$,

(point in the real world) $\mapsto$ (point on the picture).

Picture suggests: The two lines "should meet" in a point p "infinitely far away". p still can be projected to the picture.

Shortcomings of the affine spaces III

The solutions to the equation $x^2 + y^2 = 1$ form a compact geometric object. The solutions to the equation $x^2 - y^2 = 1$ however are no longer compact.



Both equations can be interpreted over the projective plane $\mathbb{P}^2$ instead of $\mathbb{A}^2$, and then everything is compact again.

Preliminary definition of projective spaces

$\mathbb{P}^n$ can be written as the union of $\mathbb{A}^n$ and the points at infinity. By definition,

$$\text{points at infinity} := {}^{\text{lines in } \mathbb{A}^n}\!\big/_{\sim},$$

where $\sim$ is the equivalence relation of lines being parallel.

Problems of this approach: Topology of $\mathbb{P}^n$ and "lines in $\mathbb{P}^n$" are hard to define.

Official definition

$\mathbb{P}^n$ is the set of 1-dimensional subspaces of the vector space K^{n+1}.
Equivalently, $\mathbb{P}^n = {K^{n+1} \setminus \{0\}}/{\sim}$, where

$$(x_0, \dots, x_n) \sim (y_0, \dots, y_n) \iff$$
$$\exists \lambda \in K \setminus \{0\} : \ (x_0, \dots, x_n) = (\lambda y_0, \dots, \lambda y_n).$$

The equivalence class of $(x_0, \dots, x_n)$ is denoted $[x_0 : \dots : x_n]$ ("homogeneous coordinates").

e.g. $[1 : 2] = [2 : 4] = [-3 : -6] \in \mathbb{P}^1(\mathbb{R})$. There is no point $[0 : 0]$.

$\mathbb{A}^n$ in $\mathbb{P}^n$

We can see $\mathbb{A}^n$ as a subset of $\mathbb{P}^n$ via the bijection

$$\begin{aligned}
&\mathbb{A}^n \to \{[x_0 : \cdots : x_n] \in \mathbb{P}^n \mid x_0 \neq 0\},\\
&(y_1, \ldots, y_n) \mapsto [1 : y_1 : \cdots : y_n],\\
&(x_1/x_0 : \cdots : x_n/x_0) \leftarrow\!\shortmid [x_0 : \cdots : x_n].
\end{aligned}$$

The points

$$\mathbb{P}^n \setminus \mathbb{A}^n = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n \mid x_0 = 0\}$$

are the "points at infinity", and they correspond to points in $\mathbb{P}^{n-1}$, or parallelism classes in $\mathbb{A}^n$.

Geometric intuition

Over the reals, we can consider the n-sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$:

$$\mathbb{S}^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_0^2 + \cdots + x_n^2 = 1\}.$$

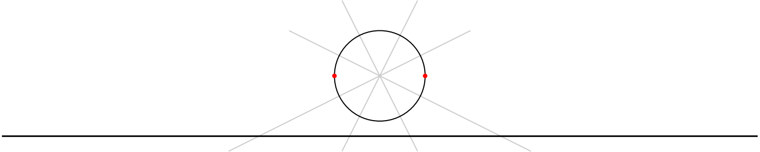
For each one-dimensional subspace $\ell \subset \mathbb{R}^{n+1}$, the intersection $\mathbb{S}^n \cap \ell$ consists of exactly two antipodal points. We thus may identify

$$\mathbb{P}^n = {}^{\mathbb{S}^n}\!/\!_{\sim},$$

where $x \sim y$ iff $x \in \{y, -y\}$.

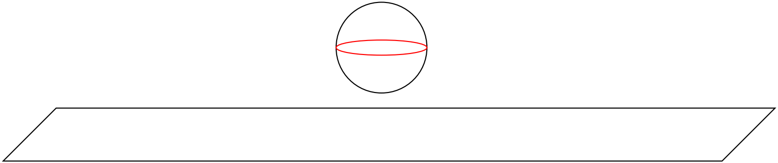
$\mathbb{P}^0$ and $\mathbb{P}^1$

$\mathbb{S}^0$ consists of two points that get identified, so $\mathbb{P}^0$ is just one point.



$\mathbb{P}^1$ can be seen as the line with an extra point at infinity added.

$\mathbb{P}^2$



Most points of $\mathbb{S}^2$ can be projected to $\mathbb{R}^2$, forming $\mathbb{A}^2 \subset \mathbb{P}^2$.
The points on the equator are can be identified with $\mathbb{P}^1$.

Miniature overview

1. Affine and projective varieties
2. Zariski topology and regular maps
3. Cones and projections
4. Nullstellensatz and applications
5. Grassmannians
6. Rational maps and Blow-Ups
7. Dimensions