Seminar Program: Examples in Algebraic Geometry

November 26, 2021

Algebraic Geometry is (at least historically) the study of geometric objects defined by polynomial equations. The easiest example of this kind is the unit circle being defined by $x^2 + y^2 = 1$. In this seminar, we will discuss basic notions and intuitions of Algebraic Geometry by studying explicit examples. We will mainly follow the first chapters of [Har92].

This seminar doesn't revolve around theorems and proofs, but rather around definitions and examples, so spend some time thoroughly discussing your examples. Each talk should take about 80–120 minutes.

Talk 1

- Define the affine and projective n-spaces $\mathbb{A}^n$ and $\mathbb{P}^n$. Find a good middle ground between the very brief treatment of [Har92, Lecture 1] and the very extensive account given in [Aud03, Chapter 5]. Try to build some geometric intuition for the spaces $\mathbb{P}^n$.
- Define the notions of affine and projective varieties, and give examples for both. Explain the notion of homegeneous polynomials and motivate why it is needed [Har92, Lecture 1].
- Introduce the standard affine charts $U_i \subset \mathbb{P}^n$.
- Discuss the examples of linear spaces, finite sets and the twisted cubic (Examples 1.1, 1.2 and 1.7).

Talk 2

- Define the Zariski topology for affine and projective varieties X [Har92, Lecture 2].
- Define the localization of rings $R_f = R[1/f]$ where R is a commutative ring and $f \in R$, e.g. following `https://en.wikipedia.org/wiki/Localization_(commutative_algebra)`.

- Introduce regular maps between affine varieties, and regular maps between quasi-projective varieties. Give an example of a non-trivial isomorphism of quasi-projective varieties $X \cong Y$.
- Introduce the Veronese and the Segre embedding and products of varieties (Examples 2.4, 2.11 and 2.21).

Talk 3

- Define cones [Har92, Example 3.1] and discuss the classification of quadrics given in Example 3.2.
- Define projections (Example 3.4).
- Define the resultant of two polynomials $f, g \in K[x_1, \dots, x_n]$ and illustrate the proofs of Theorems 3.5 and 3.12 on suitable examples. Prove Corollary 3.14.

Talk 4

- Introduce the notion of a radical ideal and explain the Nullstellensatz [Har92, Lecture 5].
- Show that the points of an affine variety X can be identified with the maximal ideals of its coordinate ring $A(X)$.
- Explain the situation for projective varieties and homogeneous ideals.
- Define the notion of an irreducible variety and illustrate Theorem 5.7 with some examples.

Talk 5

- Introduce the notions of a family of varieties and the universal hyperplane [Har92, Lecture 4].
- Define the Grassmannian and show that it is a projective variety [Har92, Lecture 6]. For this, introduce the notion of an exterior product of vector spaces and dual spaces, e.g. following [FH04, Appendix B].
- Explicitly describe the Grassmannians $G(1, n), G(n-1, n)$ and $G(n, n)$ for $n \geq 2$. Mention the explicit description of $G(2, 4)$ in [Har92, Lecture 6].
- Explain and prove Proposition 6.13, and use it to construct the join of two varieties.

Talk 6

- Explain the two definitions of "rational map" given in [Har92, Lecture 7]. Define the graph and the image of a rational map.
- Characterize the term "birationally equivalent" and show that quadric surfaces in $\mathbb{P}^3$ are rational.
- Show how to construct Blow-Ups and give examples for them.
- Tell the story of unirationality at the end of Lecture 7.

Talk 7

- Present the different characterizations of the dimension of an irreducible quasi-projective variety presented in [Har92, Lecture 11].
- Calculate the dimensions of some of the previously studied examples, as in the part "Immediate Examples".
- Explain Theorem 11.12 with suitable examples, and use it to explain Example 11.17.
- Illustrate the proof of Proposition 11.37 with a suitable example.

Talk 8

- Introduce the notion of a degree of a projective variety, and calculate it in examples [Har92, Lecture 18].
- Explain the notion of smoothness, in the extent required for the remainder of the talk.
- Introduce the various versions of Bézout's theorem. Illustrate the notion of intersection multiplicity and why it is needed with suitable examples.
- If time suffices, prove Bézout's theorem.

References

[Aud03] M. Audin. *Projective Geometry. Geometry.* Berlin, Germany: Springer, 2003, pp. 143–182.

[FH04] W. Fulton and J. Harris. *Representation Theory – A first course.* New York, NY, USA: Springer, New York, NY, 2004.

[Har92] J. Harris. *Algebraic Geometry – A first course.* New York, NY, USA: Springer-Verlag, 1992.