Combinatorics of Coxeter Groups
Solution skeches for the Exercises for June 1st

Exercise 1

Describe $T_R(u)$ for $u \in S_n^B$ (Hint: Propositions 8.1.5 and 8.1.6). Give a computable criterion for the left weak order relation $u \leq_L v$ where $u, v \in S_n^B$.

Solution sketch. The set of reflections is given by

$$T = \{(i,j)(-i,-j) \mid -j \neq i < j\} \cup \{(i,-i) \mid i > 0\}.$$

For $u \in S_n^B$, we have $(i,j)(-i,-j) \in T_R(u)$ iff $u(i) > u(j)$, and $(i,-i) \in T_R(u)$ iff $u(i) < u(-i)$, i.e. $u(i) < 0$.

The criterion for $u \leq_L v$ is given by

- for all $i, j \in [\pm n]$ with $-j \neq i < j$ such that $u(i) > u(j)$, we have $v(i) > v(j)$ and
- for all $i \in [n]$ with $u(i) < 0$, we have $v(i) < 0$.

Exercise 2

What is the window notation of the longest element $w_0 \in S_n^B$? Describe the automorphism $w \mapsto w_0 w w_0$ of S_n^B.

Solution sketch. The window notation is given by $w_0 = [-1, -2, \ldots, 1-n, -n]$. You can compute it by

- Using Proposition 8.1.2 to find the unique element $w \in S_n^B$ with $D_R(w) = S$.
- Finding the element $w \in W$ that maximizes $\mathrm{inv}_B(w)$.
- Seeing S_n^B as subset of S_{2n} and taking the longest element of S_{2n}. This is justified by the fact that S_n^B inherits the Bruhat order of S_{2n} (Corollary 8.1.9).

The automorphism is the identity, by direct computation. Alternatively, note that for $n \geq 3$, the Coxeter graph of B_n has no non-trivial automorphisms. Since $w \mapsto w_0 w w_0$ acts on W by an automorphism of the Coxter graph, this shows that the automorphism must be trivial for $n \geq 3$.

Exercise 3

Consider the following elements in S_3^B:

$$x = [-2, 3, -1], \quad u = [3, 2, 1], \quad v = [-1, 3, 2], \quad w = [2, 3, -1].$$

(a) Compute the lengths of these four elements in S_n^B.

Solution sketch. With

$$\begin{aligned}&\ell(y) = \mathrm{inv}_B(y)\\ &= |\{i,j \in \{1,2,3\} \mid i < j, y(i) > y(j)\}| + |\{i,j \in \{1,2,3\} \mid i \leq j, y(-i) > y(j)\}|,\end{aligned}$$

we get

$$\ell(x) = 1+3 = 4, \quad \ell(u) = 3+0 = 3, \quad \ell(v) = 1+1 = 2, \quad \ell(w) = 2+1 = 3.$$

(b) Show that $\ell(x) > \ell(u)$ but $x \not\geq u$ in the Bruhat order. Find $i \in \{0,1,2\}$ such that $x^J \not\geq u^J$, where $J = S \setminus \{s_i^B\}$.

Solution sketch. $\ell(x) > \ell(u)$ has been seen above. In order to show $x \not\geq u$, use Theorem 8.1.8. We have, e.g.,

$$u[1,3] = 1 > 0 = x[1,3], \quad u[-2,-2] = 2 > 1 = x[-2,-2].$$

By Corollary 2.6.2, there exists $s_i \in D_R(u)$ with $x^J \not\geq u^J$ for $J = S \setminus \{s_i\}$. So we only have to try $i = 1, 2$. The computation of these quotients is explained on page 249. We get

$$\begin{aligned}i = 0:\ & x^{\{s_1,s_2\}} = [-2,-1,3] \geq u^{\{s_1,s_2\}} = [1,2,3]\\ i = 1:\ & x^{\{s_0,s_2\}} = [2,-1,3] \not\geq u^{\{s_0,s_2\}} = [3,1,2]\\ i = 2:\ & x^{\{s_0,s_1\}} = [2,3,-1] \geq u^{\{s_0,s_1\}} = [2,3,1].\end{aligned}$$

In order to see that $[2,-1,3] \not\geq [3,1,2]$, you can observe that they are both of length 2 but not identical, so they cannot be comparable in the Bruhat order. Alternatively, use Theorem 8.1.8 again with

$$u^{\{s_0,s_1\}}[1,3] = 1 > 0 = x^{\{s_0,s_1\}}[1,3]$$

(c) It is given (and easy to see) that $v = s_0^B s_2^B$ is a reduced expression. Find a reduced expression of x that ends with $s_0^B s_2^B$. Hint: Find a reduced expression for xv^{-1}, then concatenate it with the reduced expression $v = s_0^B s_2^B$ to get an expression for x. Conclude that this expression must be reduced by either comparing lengths, or by using your criterion from Exercise 1 to show $v <_L x$.

Solution sketch. We compute $v^{-1} = v$ and

$$xv^{-1} = [-2,3,-1][-1,3,2] = [2,-1,3].$$

In order to find a reduced expression, we iteratively multiply $[2,-1,3]$ with simple reflections that make it smaller.

$$[2,-1,3] = [-1,2,3]s_1 = s_0 s_1.$$

So we conclude $x = [2, -1, 3]v = s_0 s_1 s_0 s_2$. By (a), this must be reduced. Alternatively, use Exercise 1 to compute

$$T_R(v) = \{(-1, 1),\ (2, 3)(-2, -3)\},$$
$$T_R(x) = \{(-1, 1),\ (-3, 3)\ (1, -3)(-1, 3),\ (2, 3)(-2, -3)\},$$

so that $v <_L x$.

(d) It is given that $w \leq x$ (the computation with Theorem 8.1.8 is a bit tedious). Take the reduced expression you got for x in the previous exercise, and find a subexpression that is a reduced expression for w. Show that $w \not\leq_L x$ using your criterion from Exercise 1 or a direct argument.

Solution sketch. A quick solution is to see that the window notation of $w = [2, 3, -1]$ has one negative entry, as opposed to the two negatives entries in $x = [-2, 3, -1]$. As $\ell(w) = \ell(x) - 1$, the only chance we have is to remove one of the s_0's in the reduced expression for x. Trying both possibilities shows $w = s_0 s_1 s_2$. If we had $w \leq_L x$, we would have (by comparing lengths) $sw = x$ for some simple reflection s. By comparing the number of negative entries in the window notation again, this would force $s = s_0$, which doesn't work.

Alternatively, proceed by applying the simple reflections of x to w one by one, and keeping those which make the result smaller.

$$[2, 3, -1] \leq s_0 s_1 s_0 s_2 \text{ and } [2, 3, -1]s_2 = [2, -1, 3] < [2, 3, -1]$$
$$[2, -1, 3] \leq s_0 s_1 s_0 \text{ and } [2, -1, 3]s_0 = [-2, -1, 3] > [2, -1, 3]$$
$$[2, -1, 3] \leq s_0 s_1 \text{ and } [2, -1 - 3]s_1 = [-1, 2, 3] < [2, -1, 3]$$
$$[-1, 2, 3] \leq s_0 \text{ and } [-1, 2, 3]s_0 = [1, 2, 3] < [-1, 2, 3].$$

Thus we get have $w = s_0 s_1 s_2$. With Exercise 1, we obtain

$$T_R(w) = \{(1, 3)(-1, -3),\ (2, 3)(-2, -3),\ (3, -3)\},$$

which is not a subset of $T_R(x)$. Alternatively, $xw^{-1} = [1, -2, -3]$ has length 3, which is greater than $\ell(x) - \ell(w) = 1$.