Technische Universität München
Zentrum Mathematik

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Winter term 2017/18
Exercise sheet 10

Linear algebraic groups (MA 5113)

Exercise 37 (The Frobenius endomorphism)**.** Assume that $\operatorname{char} k = p > 0$ and define the Frobenius endomorphism of GL_n by $\sigma\colon \mathrm{GL}_n \to \mathrm{GL}_n, (a_{i,j}) \mapsto (a_{i,j}^p)$.

(a) Prove that σ is is a well-defined bijective homomorphism of algebraic groups, but is not an automorphism of GL_n.

(b) Now let $G \subset \mathrm{GL}(V)$ be a σ-stable closed subgroup. Prove that the restriction $\sigma|_G\colon G \to G$ is is a bijective homomorphism of algebraic groups, but is not an automorphism of G unless G is finite.

Exercise 38 (Parameter space of maximal tori)**.** Let G be a connected linear algebraic group and let $\mathrm{Tor}(G)$ denote the the set of all maximal tori of G.

(a) Prove that the map

$$G/N_G(T) \to \mathrm{Tor}(G), gN_G(T) \mapsto g^{-1}Tg$$

is a bijection and thus defines the structure of a G-homogeneous space on $\mathrm{Tor}(G)$.

(b) Calculate explicite (homogenous) equations defining $\mathrm{Tor}(\mathrm{GL}_2)$ following the construction in 2.6.7/8. Use $V = k[x,y]_2$, the space of homogeneous polynomials of degree 2 with GL_2-action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f = f(ax+by, cx+dy)$$

and $W = k \cdot xy$.

Exercise 39 (The circle group)**.** Consider the group

$$U^1 := \{z \in \mathbb{C} : |z| = 1\}.$$

This is not a Zariski closed set in $\mathbb{C}$; but if we identify $\mathbb{C} \cong \mathbb{R}^2$, we get

$$U^1 = \mathcal{V}_{\mathbb{R}^2}(x^2 + y^2 - 1).$$

In part (a) and (b) we show that U^1 is a linear algebraic group over $\mathbb{R}$ (without defining what we mean by this). In part (c), we check that it also satisfies the characterization of a linear algebraic group as closed subgroup of GL_n.

(a) Show that the multiplication and inverse of the group U^1 is given by polynomials with coefficients in $\mathbb{R}$.

(b) Define

$$U^1_{\mathbb{C}} := \mathcal{V}_{\mathbb{C}^2}(x^2 + y^2 - 1)$$

Show that the polynomials in (a) define the structure of a linear algebraic group on $U^1_{\mathbb{C}}$.

(c) Prove that there exists an $n \in \mathbb{N}$ and an embedding $U^1 \hookrightarrow \mathrm{GL}_{n,\mathbb{R}}$ defined by polynomials.

(d) Show that $U^1_{\mathbb{C}} \cong \mathbb{G}_{m,\mathbb{C}}$, but $U^1 \ncong \mathbb{G}_{m,\mathbb{R}}$.

Exercise 40 (Permanence properties under surjective homomorphisms)**.** Let $\varphi\colon G \to H$ be a surjective homomorphism of linear algebraic groups. Prove that

(a) $\varphi(\operatorname{rad} G) = \operatorname{rad} H$.

(b) If T is maximal torus of G, then $\varphi(T)$ is a maximal torus of H.

Deadline: Friday, 14th January, 2017

If you have questions regarding the exercises, please send an email to hamacher@ma.tum.de.
The exercise classes are Fridays, 10-12 in room MI 02.08.020.
Further information about our lectures and exercises are available under

`http://www-m11.ma.tum.de/viehmann/viehmann-linear-algebraic-groups/`.

A merry christmas and a happy new year 2018 !